Math 711, Fall 2006 **Problem Set #2 Solutions**

1. If $u \in S - \mathcal{M}$ has degree 0 component u_0 that is a unit of $R/I = S_0$, and $g \in S - \{0\}$ has lowest degree nonzero component g_j, then $ug \neq 0$: in degree j it is $ug_j \neq 0$. Thus, $S - \mathcal{M}$ consists of nonzerodivisors, so that $S \subseteq S_{\mathcal{M}}$, and if $S_{\mathcal{M}}$ is an integral domain, so is S. In this case, let $r, r' \in R - \{0\}$. Choose j such that $r \in I^j - I^{j+1}$ ($\bigcap_N m^N = 0 \Rightarrow \bigcap_N I^N = 0$) and choose k such that $r' \in I^k - I^{k+1}$. Then r represents an element $v_j = r + I^{j+1} \in I^j/I^{j+1} = S_j$ that is not 0, and r' represents an element $w_k = r' + I^{k+1} \in I^k/I^{k+1} = S_k$ that is not 0, but $v_j w_k \in S_{j+k}$ is represented by rr' and is not 0. *We must have that* $rr' \in I^{j+k} - I^{j+k+1}$. Thus, R is an integral domain. $\square$

2. (a) The italicized sentence (next to last) in the solution of 1. shows that $v(rr') = v(r) + v(r')$. If $r \in I^j - I^{j+1}$ and $r' \in I^k - I^{k+1}$, then $r'' = r + r' \in I^j$ if $j \leq k$ and $r'' \in I^k$ if $k \leq j$. If say, $j < k$, then $r'' \notin I^{j+1}$ or else $r = r'' - r' \in I^{j+1} + I^k \subseteq I^{j+1}$. Similarly, $r'' \notin I^{k+1}$ if $k < j$. $\square$

(b) With $I \neq 0$, we can choose $u \in I - I^2$. Then $v(u) = 1$, and so IV must be the maximal ideal of V. Clearly, $I^n \subseteq I^nV$. But $u \in I^nV$ iff $v(u) \geq n$ and this holds for $u \in R$ iff $u \in I^n$. Since I^n is the contraction of the principal ideal I^nV, which, since V is normal, is integrally closed, it follows that I^n is integrally closed.

3. (a) This is immediate from the bonus problem in Problem Set #3.

(b) First consider the local case. Let $I = P^{(k)}$ be chosen as in part (a). Let $x \in m - P$. Then x is not a zerodivisor on any power of I, since each power of I is a symbolic power of P and so is P-primary. Then $\overline{x} = x + I \in R/I$ is not a zerodivisor in $\mathrm{gr}_I(R)$, which we know has dimension 2. Killing m kills the nonzerodivisor $\overline{x}$, and so $\dim\big(K \otimes_R \mathrm{gr}_I(R)\big) \leq \mathrm{gr}_I(R) - 1 = 2 - 1 = 1$. Thus, $\mathfrak{an}(I) \leq 1$. But I has height 1, and $\mathfrak{an}(I) = 1$. Since K is infinite, I is integral over a principal ideal. Since R is normal, principal ideals are integrally closed. Thus I is principal. In the graded case, simply observe that the least number of generators of I is the K-vector space dimension of I/mI by the homgeneous version of Nakayama's lemma, and $(I/mI)_m \cong I/mI$, since elements of $R - m$ already act invertibly on I/mI. But the K-vector space dimension of $(I/mI)_m \cong IR_m/mIR_m$ is also the least number of generators of IR_m, by the usual form of Nakayama's lemma, and we know that IR_m is principal. $\square$

4. (a) The condition is clearly necessary. To see that is sufficient, recall that R is normal if and only if every localization at a height one prime is a DVR and principal ideals are unmixed. If the first condition fails, because R_b is normal it can only fail at a height one prime containing b, which will necessarily be an associated prime of b. If the second condition fails, there will be a principal ideal cR with an associated prime Q of height at least 2. Again, since R_b is normal, Q must contain b. Since Q is an associated prime of cR, it has depth one. But then, since it contains b, it is an associated prime of b.

(b) Consider any prime Q of R. Then R_Q satisfies the same hypothesis as R, and so is normal iff its localization at every associated prime of b is normal. But these correspond to those associated primes of b in R that are contained in Q, and the condition follows.

(c) Consider any module-finite extension T of R within its fraction field $\mathcal{K}$. Since R_b is normal, it must contain T, and so is equal to T_b. It follows that the non-normal locus Y_T in $\mathrm{Spec}\,(T)$ is closed. The image X_T of Y_T in $\mathrm{Spec}\,(R)$ is closed (Y_T is a finite union of sets of the form $V(Q)$. The image of $V(Q)$ is $V(P)$, where $P = Q \cap R$, by the going-up theorem. If we take module finite extensions $R \subseteq T_1 \subseteq T_2$, then Y_{T_2} maps into Y_{T_1} (if Q_2 in T_2 lies over Q_1 in T_1 such that $(T_1)_{Q_1}$ is normal, then $(T_2)_{Q_1} = (T_1)_{Q_1}$, and so $(T_2)_{Q_2}$ must be normal.) Since closed sets have DCC (ideals have ACC), we can choose T so that X_T is minimal. We prove $X_T = \emptyset$. This means that T is normal, and so it *is* the normalization of R in $\mathcal{K}$. If $X_T \neq \emptyset$, choose $P \in X_T$. The integral closure S of R_P is module-finte over R_P, since this is true for R_m for any maximal ideal $m \supseteq P$. Choose finitely many integral fractions that span S over R_P. After we multiply by a suitable element of $R - P$, these will be integral fractions of R, and we may enlarge T by adjoining them: call the new ring W. Then W_P is normal, and so the localization of W at any prime lying over P is normal. This shows that $X_W \subseteq X_T - \{P\}$, a contradiction. $\square$

5. We know that the integral closure of a complete local domain in module-finite over it (see the second Theorem on p. 3 of the Lecture Notes of September 11), and the reduced case follows from the domain case by Problem 2 of Problem Set #1. $R[[It]] \subseteq R[[t]]$ is complete because a Cauchy sequence of elements $f_n = \sum_{j=1}^{\infty} i_{nj}t^j$ with evey $i_{nj} \in I^j$ converges: for fixed j, $\{i_{nj}\}_n$ is a Cauchy sequence of elements in I^j, and so converges to an element $i_j \in I^j$, and the sequence $\{f_n\}_n$ then converges to $\sum_{j=1}^{\infty} i_j t^j$. The quotient of $R[It]$ by $\mathfrak{A}_{h,k} = (m^h + It^k)$ and the quotient $R[[It]]/\mathfrak{A}_{h,k}R[[It]]$ are isomorphic. These quotients are local, and are therefore also isomorphic with $R[It]_{\mathcal{M}}/\mathfrak{A}^{\mathrm{e}}_{h,k}$, where $\mathfrak{B}^{\mathrm{e}}$ indicates the expansion of the ideal $\mathfrak{B}$. Since the ideals $\mathfrak{A}_{N,N}$ are cofinal with the powers $\mathcal{M}^N$ of $\mathcal{M}$, the completion of $R[It]_{\mathcal{M}}$ is $R[[It]]$, as claimed. To prove the required result, we may replace R by its normalization, and we may assume that R is a complete local normal domain, by the results sighted above in the first sentence. We may also assume $I \neq 0$. Then $R[It]$ has the same fraction field as $R[t]$. $R[t]$ is normal, and so the integral closure S of $R[It]$ in its fraction field is the same as its integral closure in $R[t]$. Thus, S is graded. If S is not module-finite over R, by successively adjoining homogeneous integral elements we get a strictly increasing chain of graded module-finite extensions of $R[It]$ within S, say $S_1 \subset S_2 \subset \cdots \subset S_j \subset \cdots$. For each j, $N_j = S_{j+1}/S_j \neq 0$ is a graded module, and so is supported on a set defined by a homogeneous ideal. Hence, $(N_j)_{\mathcal{M}} \neq 0$. Once we have localized, we have that completion is faithfully flat, and so applying $R[[It]] \otimes_{R[It]_{\mathcal{M}}} \_$ yields an infinite strictly ascending chain of integral extensions of $R[[It]]$ by integral fractions. This is a contradiction, since the normalization of $R[[It]]$ is module-finite over $R[[It]]$. $\square$

6. If $A \to B$ is a map of domains that splits as a map of A-modules and B is normal, then so is A: if $f \in A$, $g \in A - \{0\}$, and f/g is integral over A, then it is integral over B and, hence, in B, and $f = gb$. Applying the splitting ϕ, we have $f = g\phi(b)$ with $\phi(b) = a$ in A. Let $\mathcal{R} = R[It]_{ft}$. $C = \mathcal{R}_0 = R[I/f]$, and $\mathcal{R}_d = I^d C t^d$ or $(C/f^d)t^{-d}$, according as $d \geq 0$ or $d < 0$. Now, $C = R[I/f] = [R[It]_{ft}]_0 \in [S_{ft}]_0 = A$ and A splits from $B = S_{ft}$ over A, so A contains the normalization of $C = R[I/f]$. It suffices to show that A is module-finite over C. Let S_{ft} be generated by forms $\theta_1, \ldots, \theta_h$ in degrees $d_1, \ldots, d_h$. The A is $\sum_{j=1} \mathcal{R}_{-d_j}\theta_j$ and since each $\mathcal{R}_j$ is a finitely generated C-module, we are done. $\square$