Math 711, Fall 2006 **Problem Set #3**
Due: Monday, November 6

1. Let $\mathfrak{G} \subseteq I \subseteq T$ be ideals of the ring T, let $F \in T$, and let X be an indeterminate. Show that the map $\dfrac{\mathfrak{G} :_T I}{\mathfrak{G}} \to \dfrac{(\mathfrak{G}, X-F)T[X] :_{T[X]} \bigl(I,\,(X-F)\bigr)T[X]}{(\mathfrak{G}, X-F)T[X]}$ induced by the inclusion of numerator ideals is an isomorphism.[1] [Let $X - F = Y$. Reduce to the case $\mathfrak{G} = 0$.]

2. Let $T \twoheadrightarrow S$ and I be as in the construction of $W_{S/R}$, where T is finitely generated over R. Let $g_1, \ldots, g_n \in I$ be a special sequence. Let $F \in T$ map to $u \in S$. Map $T[X] \twoheadrightarrow S_u$ by sending $X \mapsto 1/u$. Show that the kernel J is $(I,\, XF-1)T$, that $g_1, \ldots, g_n,\, XF-1$ is a special sequence in J, and use this to prove that $W_{S_u/R} \cong (W_{S/R})_u$.

3. Let $R \subseteq S$ be a ring extension such that for every ideal I of R, $IS \cap R = I$. Prove that for every ideal I of R, $\overline{IS} \cap R = \overline{I}$. [It helps to use just the right characterization of when an element is in the integral closure of the ideal.]

4. If $x_1, \ldots, x_d$ is a regular sequence in the ring R and on the R-module M, prove by induction on d that $\mathrm{Tor}_i\bigl(R/(x_1, \ldots, x_d)R,\, M\bigr) = 0$ for all $i \geq 1$ using the long exact sequence for Tor coming from the short exact sequence

$$0 \to R/(x_1, \ldots, x_{d-1})R \xrightarrow{\cdot x_d} R/(x_1, \ldots, x_{d-1})R \to R/(x_1, \ldots, x_d)R \to 0.$$

5. Let (R, m, K) be a local ring. Call a reduction of I *special* if it has a minimal set of generators whose images in the degree 1 part of $K \otimes_R \mathrm{gr}_I(R)$, which is I/mI^2, are linearly independent. Show that every reduction of I has a reduction that is special. Show that there is no infinite decreasing chain of special reductions. Show that every special reduction has a special reduction that has no proper reduction. (These are related to minimal vector subspaces of I/mI^2 that generate an ideal primary to the homogeneous maximal ideal of $K \otimes_R \mathrm{gr}_I(R)$.) Hence, every ideal I has a reduction I_0 that has no proper reduction. I_0 is then called a *minimal* reduction.

6. Let notation be as in Problem 5. Show that if the smallest number of generators of I is equal to $\mathfrak{an}(I)$, then I has no proper reduction. Show that if K is infinite, then I has no proper reduction if and only if the least number of generators of I is $\mathfrak{an}(I)$.

Bonus Let S be a finitely generated $\mathbb{N}$-graded algebra with $S_0 = R$. If $d \geq 1$ is an integer, let $S\{d\} = \bigoplus_{n=0}^{\infty} S_{dn}$. Show that there is a choice of d such that $S\{d\}$ is generated over R by S_d.

[1] With $\mathfrak{G} = (g_1, \ldots, g_n)T$, this completes an omitted step in the proof of independence of $W_{S/R}$ from the finite presentation of S over R.