Math 711, Fall 2006 **Problem Set #3 Solutions**

1. Work mod $\mathfrak{G}$ and so assume $\mathfrak{G} = 0$. $T[X]$ has a T-automorphism sending X to $X - F$. Thus, $X - F$ is simply an indeterminate over T. Change notation and write X instead of $X - F$. The statement becomes that

$$0 :_T I \to \big(XT[X] :_{T[X]} (I, X)T[X]\big)/XT[X]$$

is an isomorphsim. The numerator on the right is the same as $XT[X] :_{T[X]} I$. But it is immediate that $IF \subseteq XT[X]$ if and only if I kills the constant term of F, so that $XT[X] :_{T[X]} I = (0 :_T I) + XT[X]$, and the result follows at once. $\square$

2. Let $\mathfrak{G} = (g_1, \ldots, g_)T$. Then $XF - 1$ is not a zerodivisor in $T[X]/\mathfrak{G}T[X] \cong (T/\mathfrak{G})[X]$, since its product with a nonzero element whose lowest nonzero degree term is w will have nonzero lowest degree term w.

Then $T[X]/(I,\, XF - 1)T[X] \cong S[X]/(uX - 1) \cong S_u$. Evidently, if $\mathfrak{G}$ expands to I upon localization at any minimal prime, we have that $(\mathfrak{G},\, FX - 1)T[X]$ expands to J upon localization at any minimal prime. Now consider the image of

$$M = \frac{(\mathfrak{G},\, FX-1)T[X] :_{T[X]} (I,\, FX-1)T[X]}{(\mathfrak{G},\, FX-1)T[X]} = \frac{(\mathfrak{G},\, FX-1)T[X] :_{T[X]} I}{(\mathfrak{G},\, FX-1)T[X]}$$

in $\mathcal{L}_u$, which will be $W_{S_u/R}$. Since the g_j do not involve X, the new Jacobian matrix has bottom row $(0\ 0\ \ldots\ 0\ F)$. Therefore, the new Jacobian determinant has image γu, since F maps to u. We claim that

$$(\mathfrak{G},\, FX-1)T[X] :_{T[X} I = (\mathfrak{G} :_T I)T[X] + (FX-1)T[X].$$

For the purpose of proving this we may work modulo $(FX-1)T[X]$. The equality is then seen to be equivalent to the statement that in the ring T_F we have

$$\mathfrak{G}T_F :_{T_F} IT_F = (\mathfrak{G} :_T I)T_F.$$

This is a consequence of the fact that T_F is flat over T and I is finitely generated (T is Noetherian here.) It follows that the image of M in $\mathcal{L}_u$ is the S_u-submodule generated by the images of the g_i. Each of these is the same as under the map

$$\frac{\mathfrak{G} :_T I}{\mathfrak{G}} \to \mathcal{L},$$

but multiplied by $1/u$, since we are now dividing by γu instead of by γ. Since u is invertible in S_u, the image is $(W_{S/R})_u$, as required $\square$.

3. If $r \in \overline{IS}$ then with $J = I + rR$, for some k, $(JS)^{k+1} = (IS)(JS)^k$, and $J^{k+1} \subseteq (IJ^k)S \cap R = IJ^k$. Since $IJ^k \subseteq J^{k+1}$ is obvious, we have that $J^{k+1} = IJ^k$, and so $u \in \overline{I}$. $\square$

4. Let $I = (x_1, \ldots, x_{d-1})R$ and $J = (x_1, \ldots, x_d)R$. Consider the long exact sequence $\cdots \to \mathrm{Tor}_i^R(R/I, M) \to \mathrm{Tor}_i^R(R/I, M) \to \mathrm{Tor}_i^R(R/J, M) \to \mathrm{Tor}_{i-1}(R/I, M) \to \cdots$. Then for $i > 1$ the vanishing of $\mathrm{Tor}_i(R/J, M)$ is a consequence of the fact that, by the induction hypothesis, the two surrounding terms vanish. Moreover, for $i = 1$, we have $\cdots \to \mathrm{Tor}_1^R(R/I, M) \to \mathrm{Tor}_1^R(R/J, M) \to R/I \to R/I \to R/J \to 0$. The result follows because $\mathrm{Tor}_1^R(R/I, M) = 0$ by the induction hypothesis and the map given by multiplication by x_d is injective on R/I. $\square$

5. It was shown in class that $J \subseteq I$ is a reduction if and only if the image of J in I/mI generates an ideal primary to the homgeneous maximal ideal $\mathcal{M}$ in $K \otimes \mathrm{gr}_I R$. Given any reduction J, we may choose a basis for the image of J in I/mI, and then choose elements of J that map to this basis. These elements generate a special reduction contained in J. If $J_1 \subseteq J_2$ are special reductions and their images in I/mI are the same, we claim that $J_1 = J_2$. To see this, note that since $J_1 + mI = J_2 + mI$. we can extend a set $\mathcal{S}$ of minimal generators for J_1 to a set of generators of J_2 using elements of $J_2 \cap (mI)$. This gives a minimal set $\mathcal{T}$ of generators of J_2 each of which is in $\mathcal{S}$ or is an element of mI. $\mathcal{T}$ has the same cardinality as the original set of minimal generators of J_2. $\mathcal{T}$ cannot give an image in I/mI of the correct dimension unless we have used all of the elements of $\mathcal{S}$. Since $\mathcal{T}$ has the same cardinality as the dimension of the image of J_1, it follows that $\mathcal{T} = \mathcal{S}$, and $J_2 = J_1$. Thus, if one has a strictly decreasing chain of special reductions, the images in I/mI also decrease strictly, and so the chain cannot be infinite. Given any reduction J with image V in I/mI, choose a subspace $W \subseteq V$ such that no proper subspace of W generates an $\mathcal{M}$-primary ideal. Choose elements of J mapping to a basis for W. These will generate a special reduction of I that does not properly contain any other special reduction. $\square$

6. The first statement is immediate from 5., since a reduction with $\mathfrak{an}(I)$ generators must be special, with the image in I/mI spanned by a homogeneous system of linear parameters, and cannot contain a proper reduction, since that would contain a special reduction, and the image would be too small a vector space to span a $\mathcal{M}$-primary ideal. The second statement is clear, because when K is infinite, every reduction has a reduction such that the least number of generators is $\mathfrak{an}(I)$. $\square$

BONUS Let the homogeneous generators be $F_1, \ldots, F_n$ with respective degrees $d_1, \ldots, d_n$ and let L be the least common multiple of $d_1, \ldots, d_n$. Then every monomial μ in the F_i of degree $D \geq nL$ is the product of a monomial of degree L and one of degree $D - L$: the fact that $\mu = F^{a_1} \cdots F_k^{a_k}$ has degree D implies that $\sum_{i=1}^n d_i a_i \geq nL$, and so at least one $d_i a_i \geq L$. Then we can choose $b \leq a_i$ such that $d_i b = L$, and F_i^b has degree L and is a factor of μ. If μ has degree nLh for $h > 1$ we can iterate this $n(h-1)$ times to write μ as a product of the $n(h-1)$ forms of degree L and one of degree nL. The former term can be written as a product of $h - 1$ forms of degree nL by grouping. Thus, every monomial of degree nLh is a product of h monomials of degree nL, and we may take $d = nL$. $\square$