Math 711, Fall 2006 **Problem Set #4**
Due: Wednesday, November 22

1. Find the multiplicity of the ring $K[[x^7, x^{11}, x^{13}]] \in K[[x]]$, where K is a field and x is a formal power series indeterminate.

2. Let $Y, X_1, \dots, X_n, \dots$ be countably many indeterminates over a field K. Let $R = K[Y, X_1, \dots, X_n, \dots]$. Let $I = (X_nY^n : n \geq 1)R$. Let $J = (X_1, \dots, X_n, \dots)R$. Let $S = R_Y$, which is R-flat. Show that $IS :_S JS = S$, while $(I :_R J)S = IS$. (When J is finitely generated, colon *does* commute with flat change of rings.)

3. Let M be a finitely generated module of dimension d over a local ring R of dimension d. Let $x_1, \dots, x_d$ be a system of parameters for R, and let $I = (x_1, \dots, x_d)R$. Let $n_1, \dots, n_d$ be given positive integers, let $y_i = x_i^{n_i}$, $1 \leq i \leq d$, and let $J = (y_1, \dots, y_d)R$. Is it necessarily true that $e_J(M) = (n_1 \cdots n_d)e_I(M)$? Prove your answer.

4. Let $T = K[X, Y, Z, U, V, W]$, and let $S = T/(UX + Y^2 + Z^2) = K[x, y, z, u, v, w]$. Let $m = (x, y, z, u, v, w)S$, and let $P = (x, y, z)S$, which is prime. Show that $P^{(2)} \nsubseteq m^2$.

5. (a) Let $(A, P) \subseteq (R, m)$ be an integral extension of quasilocal domains. Let t be an indeterminate, and let $F \in R[t]$ be a polynomial at least one of whose coefficients is a unit. Show that F has a multiple in $A[t]$ at least one of whose coefficients is a unit.

(b) Show that if (R, m, K) is a complete local domain, then the completion of $R(t) = R[t]_{mR[t]}$, where t is an indeterminate, is of pure dimension. [You may assume that R is module-finite over A regular. Then $R \hookrightarrow A^{\oplus h}$ for some h, and $R \otimes_A A(t) \hookrightarrow A(t)^{\oplus h}$. Show that $R \otimes_A A(t) \cong R(t)$ using (a), and also use that $A(t)$ is regular.]

6. Let $r, s \geq 1$ be integers and let $X_1, \dots, X_r, Y_1, \dots, Y_s$ be indeterminates over the field K. Let $S = K[X_iY_j : 1 \leq i \leq r,\ 1 \leq j \leq s] \subseteq K[X_1, \dots, X_r, Y_1, \dots, Y_s]$, and let R be the localization of S at the maximal ideal generated by all the X_iY_j. What is the multiplicity of R?

BONUS Let $0 < a_1 < \cdots < a_k$ be integers whose greatest common divisor is 1. (They need not be relatively prime in pairs.) Generalize problem 1. by finding the multiplicitiy of the ring $K[[t^{a_1}, \dots, t^{a_k}]] \subseteq K[[t]]$