Math 711, Fall 2006 **Problem Set #4 Solutions**

1. Call the ring R. As in the solution of the Bonus Problem below, $R' = K[[x]]$ and x^7R is a reduction if m. Then $e(R) = e_{x^7R}(R) = \ell(R/x^7R)$. The monomials with exponents $0, 7, 11, 13, 14, 18, 20, 21, 22, 24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36, 37, 38, \ldots$ (and all those with exponent > 38) are precisely the ones that are in R, and those with exponents $7, 14, 18, 20, 21, 25, 27, 28, 29, 31, 32, 33, 34, 35, 36, 38, \ldots$ (and all with exponent > 38) are in x^7R. Thus, the quotient has a K-basis consisting of the 7 monomials with exponents $0, 11, 13, 22, 24, 26, 37$, and $e(R) = 7$. The Bonus Problem solution uses a different method.

2. It is clear that $IS = JS$, since Y is a unit in S, and so it will suffice to show that $I :_R J = I$. Clearly, $I :_R J$ is generated by monomials. Suppose, to the contrary, it contains a monomial $\mu \in Y^nR - Y^{n+1}R$, and $\mu \notin X_iR$ for $i \leq n$. Then $\mu X_{n+1} \notin J$, a contradicition. $\square$

3. Yes, since by Lech's Theorem $e_J(M) = \lim\limits_{k\to\infty} \dfrac{\ell\big(M/\big((x_1^{n_1})^k, \, \ldots \, , (x_d^{n_d})^k\big)M\big)}{k^d}$ and we may rewrite this as $n_1 \cdots n_d \lim\limits_{k\to\infty} \dfrac{\ell\big(M/(x_1^{kn_1}, \, \ldots \, , x_d^{kn_d})M\big)}{(kn_1)\cdots(kn_d)} = n_1 \cdots n_d \, e_I(M)$. $\square$

4. Since $u \notin P$ while $ux = -y^2 - z^2 \in P^2$, we have that $x \in P^{(2)}$. Clearly, $x \notin m^2$. $\square$

5. (a) Since $(A, P, K) \hookrightarrow (R, m, L)$ is integral, so are $K \to L$ and $K \to R/PR$. Thus, $\dim(R/PR) = 0$, and $m \subseteq \mathrm{Rad}\,(PR)$. If the result is false, $FR[t] \cap A[t] \subseteq PA[t]$. Then $A[t]/(FR[t] \cap A[t]) \hookrightarrow R[t]/FR[t]$ is integral, and $\mathcal{P} = PA[t]/(FR[t] \cap A[t])$ is prime in $A[t]/(FR[t] \cap A[t])$, and so lies under a prime ideal $\mathcal{M}$ of $R[t]/FR[t]$. Let $\overline{F}$ be the image of F mod $mR[t]$. Since $m \subseteq \mathrm{Rad}\,(PR)$, $m \subseteq \mathcal{M}$, and so $K[t]$ injects into a quotient of $L[t]/\overline{F}(t)$, which is integral over L and, hence, over K, a contradiction. $\square$

(b) Let $r_1, \, \ldots \, , r_h \in R$ be a maximal set linearly independent over A, so that $G = Ar_1 + \cdots + Ar_h$ is A-free. R/G must be A-torsion: if $r \in R$ were non-torsion mod G, we would have $Ar \cap G = 0 \Rightarrow r_1, \, \ldots \, , r_j, \, r$ are linearly independent over A. Since R is module-finite over A, there exists $c \in A - \{0\}$ such that $R \cong aR \subseteq G$. By part (a), localizing at $A[t] - PA[t]$ inverts all elements of $R - mR[t]$, and so $R(t) \cong R \otimes_A A(t) \hookrightarrow A(t)^{\oplus h}$ and $\widehat{R(t)} \hookrightarrow \widehat{A(t)}^{\oplus h}$. Since $\widehat{A(t)}$ is regular, $\widehat{A(t)}^{\oplus h}$ has pure dimension, and so does $\widehat{R(t)}$. $\square$

6. The associated graded ring of R with respect to its maximal ideal is S, graded so that monomials of degree $2n$ (degree n in both the x_i and the y_j) span the nth graded piece. Thus $\dim(S_n) = \binom{n+r-1}{r-1}\binom{n+s-1}{s-1}$, which has degree $r + s - 2$, and the leading coefficient is $\dfrac{1}{(r-1)!\,(s-1)!}$, so that $e(R) = \dfrac{(r+s-2)!}{(r-1)!\,(s-1)!} = \dbinom{r+s-2}{r-1}$.

BONUS Call the ring R. Since the a_j are relatively prime, $t \in \mathrm{frac}\,(R)$. (We can write $1 = \sum_j n_j a_j$ where the $n_j \in \mathbb{Z}$, and then $t = \prod_j (t^{a_j})^{n_j}$.) Then $t \in R'$, the normalization of R, since t satisfies $z^{a_1} - t^{a_1} = 0$. Hence, $R' = K[[t]]$, and, with $x = t^{a_1}$, xR is a minimal reduction of m, since $t^{a_i} \in t^{a_1}K[[t]]$ for all i, and so every $t^{a_i} \in \overline{t^{a_1}R}$. Thus, $e(R) = e_x(R) = \mathrm{rank}_{K[[x]]}R = [K((t)) : K((t^{a_1}))] = a_1$. $\square$