Math 711, Fall 2006 **Problem Set #5**
Due: Monday, December 11

1. Let R and S be finitely generated K-algebras with $P \subseteq R$ and $Q \subseteq S$ maximal ideals such that $K \hookrightarrow R \twoheadrightarrow R/P$ and $K \hookrightarrow S \twoheadrightarrow S/Q$ are isomorphisms. Let $A = R_P$, $B = S_Q$, and let $C = T_{\mathcal{M}}$ where $T = A \otimes_K B$ and $\mathcal{M} = PT + QT$. Let M and N be finitely generated modules over A and B, respectively. Let $W = (M \otimes_K N)_{\mathcal{M}}$. Prove or disprove:
(a) $\nu(W) = \nu(M)\nu(N)$.
(b) If M and N are maximal Cohen-Macaulay modules, then so is W, and, in this case
(1) $e(W) = e(M)e(N)$, and
(2) if M and N are linear maximal Cohen-Macaulay modules, so is W.

2. Consider an $r \times s$ matrix of indeterminates $x_{i,j}$ over a field K, where $1 \leq r \leq s$. Let $T = K[x_{ij} : 1 \leq i \leq r,\ 1 \leq j \leq s]$. Consider the $s-r+1$ diagonals D_j, $1 \leq j \leq s-r+1$, in X where D_j consists of the r elements $x_{1,j},\ x_{2,\,j+1},\ x_{1+k,\,j+k},\ \ldots,\ x_{r,\,j+r-1}$. Let I be the ideal of T generated by $I_r(X)$, by all of the variables not occurring on any of the D_j, and for every j by the differences $x_{1+k,j+k} - x_{1,j}$, $1 \leq k \leq r-1$, so that all of the elements on D_j are equal to $x_j = x_{1,j}$ in T/I. Prove that $T/I \cong K[x_1, \ldots, x_{s-r+1}]/m^r$, where $m = (x_1, \ldots, x_{s-r+1})$.

3. Let R be a Cohen-Macaulay standard graded K-algebra, where K is a field. That is, R is a finitely generated $\mathbb{N}$-graded algebra over $R_0 = K$ that is generated by its 1-forms R_1. Suppose that $\dim(R) = d$
(a) Let $f_1, \ldots, f_d$ be a homgeneous system of parameters consisting of forms with positive degrees $k_1, \ldots, k_d$ respectively. Let h denote the largest degree of a nonzero homogeneous element of $R/(f_1, \ldots, f_d)R$. Prove that $h - (k_1 + \cdots + k_d)$ is independent of the choice of the homogeneous system of parameters $f_1, \ldots, f_d$. [The Hilbert-Poincare series $\sum_{n=0}^{\infty} \dim_K(R_i)z^n$ of R can be written as a rational function with denominator $(1-z^{k_1})\cdots(1-z^{k_n})$. What is its degree?] The value of $h - (k_1 + \cdots + k_h)$ is called the $\mathbf{a}$-*invariant* of R, and is denoted $\mathbf{a}(R)$.
(b) Show that the Segre product $R \circledS_K K[s,t]$ is Cohen-Macaulay iff $\mathbf{a}(R) < 0$.

4. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules over the local ring (R, m, K) such that M and M'' are maximal Cohen-Macaulay modules and $M' \neq 0$.
(a) Show that M' is a maximal Cohen-Macaulay module.
(b) Show that if M and M'' are linear maximal Cohen-Macaulay modules, then so is M'.

5. Let X_{ij} be rs indeterminates over a field K, where $1 \leq r \leq s$, and let $R = K[X_{ij}]/I = K[x_{ij}]$, where $I = I_2\big(X_{ij}\big)$. Partially order the x_{ij}, so that $x_{ij} \leq x_{rs}$ precisely if $i \leq r$ and $j \leq s$. Let f be the K-algebra map $R \to S = K[U_1, \ldots, U_r] \circledS_K K[V_1, \ldots, V_s]$, where the U_i and V_j are new indeterminates, such that $x_{ij} \mapsto U_i V_J$. (Why is this well-defined?) Show that the set $\mathcal{S}$ of monomials in the x_{ij} such that the set of variables occurring is linearly ordered spans R as a K-vector space. Show that f maps $\mathcal{S}$ bijectively onto the monomials in S. Conclude that f is an isomorphism.

6. (R, m, K) is a complete Cohen-Macaulay local domain of char. $p > 0$, with K perfect, and $\lim_{e\to\infty}\big(\ell(R/m^{[p^e]})/p^{e\dim(R)}\big) = e(R)$. Prove: if $R \to S$ is flat local, then $e(R) \leq e(S)$.