Math 711: Lecture of September 5, 2007

Throughout these lecture notes all given rings are assumed commutative, associative, with identity and modules are assumed unital. Homomorphisms are assumed to preserve the identity. With a few exceptions that will be noted as they occur, given rings are assumed to be Noetherian. However, we usually include this hypothesis, especially in formal statements of theorems.

Our objective is to discuss tight closure closure theory and its connection with the existence of big Cohen-Macaulay algebras, as well as the applications that each of these have: they have many in common.

At certain points in these notes we will include material not covered in class that we want to assume. We indicate where such digressions begin and end with double bars before and after, just as we have done for these two paragraphs. On first perusal, the reader may wish to read only the unfamiliar definitions and the statements of theorems given, and come back to the proofs later.

In particular, the write-up of this first lecture is much longer than will be usual, since a substantial amount of prerequisite material is explained, often in detail, in this manner.

By a *quasilocal ring* (R, m, K) we mean a ring with a unique maximal ideal m: in this notation, $K = R/m$. A quasilocal ring is called *local* if it is Noetherian. A homomorphism $h : R \to S$ from a quasilocal ring (R, m, K) to a quasilocal ring (S, m_S, K_S) is called *local* if $h(m) \subseteq m_S$, and then h induces a map of residue fields $K \to K_S$.

If $x_1, \ldots, x_n \in R$ and M is an R-module, the sequence $x_1, \ldots, x_n$ is called a *possibly improper* regular sequence on M if x_1 is not a zerodivisor on M and for all i, $0 \leq i \leq n-1$, x_{i+1} is not a zerodivisor on $M/(x_1, \ldots, x_i)M$. A possibly improper regular sequence is called a *regular sequence* on M if, in addition, $(*)$ $(x_1, \ldots, x_n)M \neq M$. When $(*)$ fails, the regular sequence is called *improper*. When $(*)$ holds we may say that the regular sequence is *proper* for emphasis, but this use of the word "proper" is not necessary.

Note that every sequence of elements is an improper regular sequence on the 0 module, and that a sequence of any length consisting of the element 1 (or units of the ring) is an improper regular sequence on every module.

If $x_1, \ldots, x_n \in m$, the maximal ideal of a local ring (R, m, K), and M is a nonzero finitely generated R-module, then it is automatic that if $x_1, \ldots, x_n$ is a possibly improper regular sequence on M then $x_1, \ldots, x_n$ is a regular sequence on M: we know that $mM \neq M$ by Nakayama's Lemma.

If $x_1, \ldots, x_n \in R$ is a possibly improper regular sequence on M and and S is any flat R-algebra, then the images of $x_1, \ldots, x_n$ in S form a possibly improper regular sequence on $S \otimes_R M$. By a straightforward induction on n, this reduces to the case where $n = 1$, where it follows from the observation that if $0 \to M \to M$ is exact, where the map is given by multiplication by x, this remains true when we apply $S \otimes_R \_$. In particular, this holds when S is a localization of R.

If $x_1, \ldots, x_n$ is a regular sequence on M and S is flat over R, it remains a regular sequence provided that $S \otimes_R \left(M/(x_1, \ldots, x_n)M \right) \neq 0$, which is always the case when S is faithfully flat over R.

Nakayama's Lemma, including the homogeneous case

Recall that in Nakayama's Lemma one has a *finitely generated module M* over a quasilocal ring (R, m, K). The lemma states that if $M = mM$ then $M = 0$. (In fact, if $u_1, \ldots, u_h$ is a set of generators of M with h minimum, the fact that $M = mM$ implies that $M = mu_1 + \cdots mu_h$. In particular, $u_h = f_1 u_1 + \cdots + f_h u_h$, and so $(1 - f_h)u_h = f_1 u_1 + \cdots + f_{h-1} u_{h-1}$ (or 0 if $h = 1$). Since $1 - f_h$ is a unit, u_h is not needed as a generator, a contradiction unless $h = 0$.)

By applying this result to M/N, one can conclude that if M is finitely generated (or finitely generated over N), and $M = N + mM$, then $M = N$. In particular, elements of M whose images generate M/mM generate M: if N is the module they generate, we have $M = N + mM$. Less familiar is the homogeneous form of the Lemma: it does not need M to be finitely generated, although there can be only finitely many negative graded components (the detailed statement is given below).

First recall that if H is an additive semigroup with 0 and R is an H-graded ring, we also have the notion of an H-graded R-module M: M has a direct sum decomposition

$$M = \bigoplus_{h \in H} M_h$$

as an abelian group such that for all $h, k \in H$, $R_h M_k \subseteq M_{h+k}$. Thus, every M_h is an R_0-module. A submodule N of M is called graded (or homogeneous) if

$$N = \bigoplus_{h \in H} (N \cap M_h).$$

An equivalent statement is that the homogeneous components in M of every element of N are in N, and another is that N is generated by forms of M.

Note that if we have a subsemigroup $H \subseteq H'$, then any H-graded ring or module can be viewed as an H'-graded ring or module by letting the components corresponding to elements of $H' - H$ be zero.

In particular, an $\mathbb{N}$-graded ring is also $\mathbb{Z}$-graded, and it makes sense to consider a $\mathbb{Z}$-graded module over an $\mathbb{N}$-graded ring.

Nakayama's Lemma, homogeneous form. *Let R be an $\mathbb{N}$-graded ring and let M be any $\mathbb{Z}$-graded module such that $M_{-n} = 0$ for all sufficiently large n (i.e., M has only finitely many nonzero negative components). Let I be the ideal of R generated by elements of positive degree. If $M = IM$, then $M = 0$. Hence, if N is a graded submodule such that $M = N + IM$, then $N = M$, and a homogeneous set of generators for M/IM generates M.*

Proof. If $M = IM$ and $u \in M$ is nonzero homogeneous of smallest degree d, then u is a sum of products $i_t v_t$ where each $i_t \in I$ has positive degree, and every v_t is homogeneous, necessarily of degree $\geq d$. Since every term $i_t v_t$ has degree strictly larger than d, this is a contradiction. The final two statements follow exactly as in the case of the usual form of Nakayama's Lemma. $\square$

In general, regular sequences are not permutable: in the polynomial ring $R = K[x, y, z]$ over the field K, $x - 1, xy, xz$ is a regular sequence but $xy, xz, x - 1$ is not. However, if M is a finitely generated nonzero module over a local ring (R, m, K), a regular sequence on M is permutable. This is also true if R is $\mathbb{N}$-graded, M is $\mathbb{Z}$-graded but nonzero in only finitely many negative degrees, and the elements of the regular sequence in R have positive degree.

To see why, note that we get all permutations if we can transpose two consecutive terms of a regular sequence. If we kill the ideal generated by the preceding terms times the module, we come down to the case where we are transposing the first two terms. Since the ideal generated by these two terms does not depend on their order, it suffices to consider the case of regular sequences x, y of length 2. The key point is to prove that y is not a zerodivisor on M. Let $N \subseteq M$ by the annihilator of y. If $u \in N$, $yu = 0 \in xM$ implies that $u \in xM$, so that $u = xv$. Then $y(xv) = 0$, and x is not a zerodivsior on M, so that $yv = 0$, and $v \in N$. This shows that $N = xN$, contradicting Nakayama's Lemma (the local version or the homogeneous version, whichever is appropriate).

The next part of the argument does not need the local or graded hypothesis: it works quite generally. We need to show that x is a nonzerodivisor on M/yM. Suppose that $xu = yv$. Since y is a nonzerodivisor on xM, we have that $v = xw$, and $xu = yxw$. Thus $x(u - yw) = 0$. Since x is a nonzerodivisor on M, we have that $u = yw$, as required. $\square$

The Krull dimension of a ring R may be characterized as the supremum of lengths of chains of prime ideals of R, where the length of the strictly ascending chain

$$P_0 \subset P_1 \subset \cdots \subset P_n$$

is n. The Krull dimension of the local ring (R, m, K) may also be characterized as the least integer n such that there exists a sequence $x_1, \ldots, x_n \in m$ such that $m =$

Rad $\big((x_1, \ldots, x_n)R\big)$ (equivalently, such that $\overline{R} = R/(x_1, \ldots, x_n)R$ is a zero-dimensional local ring, which means that $\overline{R}$ is an Artinian local ring).

Such a sequence is called a *system of parameters* for R.

One can always construct a system of parameters for the local ring (R, m, K) as follows. If $\dim(R) = 0$ the system is empty. Otherwise, the maximal ideal cannot be contained in the union of the minimal primes of R. Choose $x_1 \in m$ not in any minimal prime of R. In fact, it suffices to choose x_1 not in any minimal primes P such that $\dim(R/P) = \dim(R)$. Once $x_1, \ldots, x_k$ have been chosen so that $x_1, \ldots, x_k$ is part of a system of parameters (equivalently, such that $\dim\big(R/(x_1, \ldots, x_k)R\big) = \dim(R) - k)$), if $k < \dim(R)$ the minimal primes of $(x_1, \ldots, x_k)R$ cannot cover m. It follows that we can choose x_{k+1} not in any such minimal prime, and then $x_1, \ldots, x_{k+1}$ is part of a system of parameters. By induction, we eventually reach a system of parameters for R. Notices that in choosing x_{k+1}, it actually suffices to avoid only those minimal primes Q of $(x_1, \ldots, x_k)R$ such that $\dim(R/Q) = \dim\big(R/(x_1, \ldots, x_k)R\big)$ (which is $\dim(R) - k$).

A local ring is called *Cohen-Macaulay* if some (equivalently, every) system of parameters is a regular sequence on R. These include regular local rings: if one has a minimal set of generators of the maximal ideal, the quotient by each in turn is again regular and so is a domain, and hence every element is a nonzerodivisor modulo the ideal generated by its predecessors. Moreover, *local complete intersections*, i.e., local rings of the form $R/(f_1, \ldots, f_h)$ where R is regular and $f_1, \ldots, f_h$ is part of a system of parameters for R, are Cohen-Macaulay. It is quite easy to see that if R is Cohen-Macaulay, so is R/I whenever I is generated by a regular sequence.

If R is a Cohen-Macaulay local ring, the localization of R at any prime ideal is Cohen-Macaulay. We define an arbitrary Noetherian ring to be *Cohen-Macaulay* if all of its local rings at maximal ideals (equivalently, at prime ideals) are Cohen-Macaulay.

Cohen-Macaulay rings in the graded and local cases

We want to put special emphasis on the graded case for several reasons. One is its importance in projective geometry. Beyond that, there are many theorems about the graded case that make it easier both to understand and to do calculations. Moreover, many of the most important examples of Cohen-Macaulay rings are graded.

We first note:

Proposition. *Let M be an $\mathbb{N}$-graded or $\mathbb{Z}$-graded module over an $\mathbb{N}$-graded or $\mathbb{Z}$-graded Noetherian ring S. Then every associated prime of M is homogeneous. Hence, every*

minimal prime of the support of M is homogeneous and, in particular the associated (hence, the minimal) primes of S are homogeneous.

Proof. Any associated prime P of M is the annihilator of some element u of M, and then every nonzero multiple of $u \neq 0$ can be thought of as a nonzero element of $S/P \cong Su \subseteq M$, and so has annihilator P as well. If u_i is a nonzero homogeneous component of u of degree i, its annihilator J_i is easily seen to be a homogeneous ideal of S. If $J_h \neq J_i$ we can choose a form F in one and not the other, and then Fu is nonzero with fewer homgeneous components then u. Thus, the homogeneous ideals J_i are all equal to, say, J, and clearly $J \subseteq P$. Suppose that $s \in P - J$ and subtract off all components of S that are in J, so that no nonzero component is in J. Let $s_a \notin J$ be the lowest degree component of s and u_b be the lowest degree component in u. Then $s_a u_b$ is the only term of degree $a + b$ occurring in $su = 0$, and so must be 0. But then $s_a \in \mathrm{Ann}_S u_b = J_b = J$, a contradiction. $\square$

Corollary. *Let K be a field and let R be a finitely generated $\mathbb{N}$-graded K-algebra with $R_0 = K$. Let $\mathcal{M} = \bigoplus_{d=1}^{\infty} R_j$ be the homogeneous maximal ideal of R. Then $\dim(R) =$ height $(\mathcal{M}) = \dim(R_{\mathcal{M}})$.*

Proof. The dimension of R will be equal to the dimension of R/P for one of the minimal primes P of R. Since P is minimal, it is an associated prime and therefore is homogenous. Hence, $P \subseteq \mathcal{M}$. The domain R/P is finitely generated over K, and therefore its dimension is equal to the height of every maximal ideal including, in particular, $\mathcal{M}/P$. Thus,

$$\dim(R) = \dim(R/P) = \dim\big((R/P)_{\mathcal{M}}\big) \leq \dim R_{\mathcal{M}} \leq \dim(R),$$

and so equality holds throughout, as required. $\square$

Proposition (homogeneous prime avoidance). *Let R be an $\mathbb{N}$-graded algebra, and let I be a homogeneous ideal of R whose homogeneous elements have positive degree. Let $P_1, \ldots, P_k$ be prime ideals of R. Suppose that every homogeneous element $f \in I$ is in $\bigcup_{i=1}^{k} P_i$. Then $I \subseteq P_j$ for some j, $1 \leq j \leq k$.*

Proof. We have that the set H of homogeneous elements of I is contained in $\bigcup_{i=1}^{k} P_k$. If $k = 1$ we can conclude that $I \subseteq P_1$. We use induction on k. Without loss of generality, we may assume that H is not contained in the union of any $k - 1$ if the P_j. Hence, for every i there is a homogeous element $g_i \in I$ that is not in any of the P_j for $j \neq i$, and so it must be in P_i. We shall show that if $k > 1$ we have a contradiction. By raising the g_i to suitable positive powers we may assume that they all have the same degree. Then $g_1^{k-1} + g_2 \cdots g_k \in I$ is a homogeneous element of I that is not in any of the P_j: g_1 is not in P_j for $j > 1$ but is in P_1, and $g_2 \cdots g_k$ is in each of $P_2, \ldots, P_k$ but is not in P_1. $\square$

Now suppose that R is a finitely generated $\mathbb{N}$-graded algebra over $R_0 = K$, where K is a field. By a *homogenous system of parameters* for R we mean a sequence of homogeneous elements $F_1, \ldots, F_n$ of positive degree in R such that $n = \dim(R)$ and $R/F_1, \ldots, F_n)$ has Krull dimension 0. When R is a such a graded ring, a homogeneous system of parameters

always exists. By homogeneous prime avoidance, there is a form F_1 that is not in the union of the minimal primes of R. Then $\dim(R/F_1) = \dim(R) - 1$. For the inductive step, choose forms of positive degree $F_2, \ldots, F_n$ whose images in R/F_1R are a homogeneous system of parameters for R/F_1R. Then $F_1, \ldots, F_n$ is a homogeneous system of parameters for R. $\square$

Moreover, we have:

Theorem. *Let R be a finitely generated $\mathbb{N}$-graded K-algebra with $R_0 = K$ such that $\dim(R) = n$. A homogeneous system of parameters $F_1, \ldots, F_n$ for R always exists. Moreover, if $F_1, \ldots, F_n$ is a sequence of homogeneous elements of positive degree, then the following statements are equivalent.*

(1) *$F_1, \ldots, F_n$ is a homogeneous system of parameters.*

(2) *m is nilpotent modulo $(F_1, \ldots, F_n)R$.*

(3) *$R/(F_1, \ldots, F_n)R$ is finite-dimensional as a K-vector space.*

(4) *R is module-finite over the subring $K[F_1, \ldots, F_n]$.*

Moreover, when these conditions hold, $F_1, \ldots, F_n$ are algebraically independent over K, so that $K[F_1, \ldots, F_n]$ is a polynomial ring.

Proof. We have already shown existence.

$(1) \Rightarrow (2)$. If $F_1, \ldots, F_n$ is a homogeneous system of parameters, we have that

$$\dim\big(R/F_1, \ldots, F_n)\big) = 0.$$

We then know that all prime ideals are maximal. But we know as well that the maximal ideals are also minimal primes, and so must be homogeneous. Since there is only one homogenous maximal ideal, it must be $m/(F_1, \ldots, F_n)R$, and it follows that m is nilpotent on $(F_1, \ldots, F_n)R$.

$(2) \Rightarrow (3)$. If m is nilpotent modulo $(F_1, \ldots, F_n)R$, then the homogeneous maximal ideal of $\overline{R} = R/(F_1, \ldots, F_n)R$ is nilpotent, and it follows that $[\overline{R}]_d = 0$ for all $d \gg 0$. Since each $\overline{R}_d$ is a finite dimensional vector space over K, it follows that $\overline{R}$ itself is finite-dimensional as a K-vector space.

$(3) \Rightarrow (4)$. This is immediate from the homogeneous form of Nakayama's Lemma: a finite set of homogeneous elements of R whose images in $\overline{R}$ are a K-vector space basis will span R over $K[F_1, \ldots, F_n]$, since the homogenous maximal ideal of $K[F_1, \ldots, F_n]$ is generated by $F_1, \ldots, F_n$.

$(4) \Rightarrow (1)$. If R is module-finite over $K[F_1, \ldots, F_n]$, this is preserved mod $(F_1, \ldots, F_n)$, so that $R/(F_1, \ldots, F_n)$ is module-finite over K, and therefore zero-dimensional as a ring.

Finally, when R is a module-finite extension of $K[F_1, \ldots, F_n]$, the two rings have the same dimension. Since $K[F_1, \ldots, F_n]$ has dimension n, the elements $F_1, \ldots, F_n$ must be algebraically independent. $\square$

The technique described in the discussion that follows is very useful both in the local and graded cases.

Discussion: making a transition from one system of parameters to another. Let R be a Noetherian ring of Krull dimension n, and assume that one of the two situations described below holds.

(1) (R, m, K) is local and $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are two systems of parameters.

(2) R is finitely generated $\mathbb{N}$-graded over $R_0 = K$, a field, m is the homogeneous maximal ideal, and $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are two homogeneous systems of parameters for R.

We want to observe that in this situation there is a finite sequence of systems of parameters (respectively, homogeneous systems of parameters in case (2)) starting with $f_1, \ldots, f_n$ and ending with $g_1, \ldots, g_n$ such that any two consecutive elements of the sequence agree in all but one element (i.e., after reordering, only the ith terms are possibly different for a single value of i, $1 \leq i \leq n$). We can see this by induction on n. If $n = 1$ there is nothing to prove. If $n > 1$, first note that we can choose h (homogeneous of positive degree in the graded case) so as to avoid all minimal primes of $(f_2, \ldots, f_n)R$ and all minimal primes of $(g_2, \ldots, g_n)R$. Then it suffices to get a sequence from $h, f_2, \ldots, f_n$ to $h, g_2, \ldots, g_n$, since the former differs from $f_1, \ldots, f_n$ in only one term and the latter differs from $g_1, \ldots, g_n$ in only one term. But this problem can be solved by working in R/hR and getting a sequence from the images of $f_2, \ldots, f_n$ to the images of $g_2, \ldots, g_n$, which we can do by the induction hypothesis. We lift all of the systems of parameters back to R by taking, for each one, h and inverse images of the elements in the sequence in R (taking a homogeneous inverse image in the graded case), and always taking the same inverse image for each element of R/hR that occurs. $\square$

The following result now justifies several assertions about Cohen-Macaulay rings made without proof earlier.

Note that a regular sequence in the maximal ideal of a local ring (R, m, K) is always part of a system of parameters: each element is not in any associated prime of the ideal generated by its predecessors, and so cannot be any minimal primes of that ideal. It follows that as we kill successive elements of the sequence, the dimension of the quotient drops by one at every step.

Corollary. *Let (R, m, K) be a local ring. There exists a system of parameters that is a regular sequence if and only if every system of parameters is a regular sequence. In this case, for every prime ideal I of R of height k, there is a regular sequence of length k in I.*

Moreover, for every prime ideal P of R, R_P also has the property that every system of parameters is a regular sequence.

Proof. For the first statement, we can choose a chain as in the comparison statement just above. Thus, we can reduce to the case where the two systems of parameters differ in only

one element. Because systems of parameters are permutable and regular sequences are permutable in the local case, we may assume that the two systems agree except possibly for the last element. We may therefore kill the first $\dim(R) - 1$ elements, and so reduce to the case where x and y are one element systems of parameters in a local ring R of dimension 1. Then x has a power that is a multiple of y, say $x^h = uy$, and y has a power that is a multiple of x. If x is not a zerodivisor, neither is x^h, and it follows that y is not a zerodivisor. The converse is exactly similar.

Now suppose that I is any ideal of height h. Choose a maximal sequence of elements (it might be empty) of I that is part of a system of parameters, say $x_1, \ldots, x_k$. If $k < h$, then I cannot be contained in the union of the minimal primes of $(x_1, \ldots, x_k)$: otherwise, it will be contained in one of them, say Q, and the height of Q is bounded by k. Chose $x_{k+1} \in I$ not in any minimal prime of $(x_1, \ldots, x_k)R$. Then $x_1, \ldots, x_{k+1}$ is part of a system of parameters for R, contradicting the maximality of the sequence $x_1, \ldots, x_k$.

Finally, consider the case where $I = P$ is prime. Then P contains a regular sequence $x_1, \ldots, x_k$, which must also be regular in R_P, and, hence, part of a system of parameters. Since $\dim(R_P) = k$, it must be a system of parameters. $\square$

Lemma. *Let K be a field and assume either that*

(1) *R is a regular local ring of dimension n and $x_1, \ldots, x_n$ is a system of parameters*

  or

(2) *$R = K[x_1, \ldots, x_n]$ is a graded polynomial ring over K in which each of the x_i is a form of positive degree.*

Let M be a nonzero finitely generated R-module which is $\mathbb{Z}$-graded in case (2). Then M is free if and only if $x_1, \ldots, x_n$ is a regular sequence on M.

Proof. The "only if" part is clear, since $x_1, \ldots, x_n$ is a regular sequence on R and M is a direct sum of copies of R. Let $m = (x_1, \ldots, x_n)R$. Then $V = M/mM$ is a finite-dimensional K-vector space that is graded in case (2). Choose a K-vector space basis for V consisting of homogeneous elements in case (2), and let $u_1, \ldots, u_h \in M$ be elements of M that lift these basis elements and are homogeneous in case (2). Then the u_j span M by the relevant form of Nakayama's Lemma, and it suffices to prove that they have no nonzero relations over R. We use induction on n. The result is clear if $n = 0$.

Assume $n > 0$ and let $N = \{(r_1, \ldots, r_h) \in R^h : r_1 u_1 + \cdots + r_h u_h = 0\}$. By the induction hypothesis, the images of the u_j in $M/x_1 M$ are a free basis for $M/x_1 M$. It follow that if $\rho = (r_1, \ldots, r_h) \in N$, then every r_j is 0 in $R/x_1 R$, i.e., that we can write $r_j = x_1 s_j$ for all j. Then $x_1(s_1 u_1 + \cdots + s_h u_h) = 0$, and since x_1 is not a zerodivisor on M, we have that $s_1 u_1 + \cdots + s_h u_h = 0$, i.e., that $\sigma = (s_1, \ldots, s_h) \in N$. Then $\rho = x_1 \sigma \in x_1 N$, which shows that $N = x_1 N$. Thus, $N = 0$ by the appropriate form of Nakayama's Lemma. $\square$

We next observe:

Theorem. *Let R be a finitely generated graded algebra of dimension n over $R_0 = K$, a field. Let m denote the homogeneous maximal ideal of R. The following conditions are equivalent.*

(1) *Some homogeneous system of parameters is a regular sequence.*

(2) *Every homogeneous system of parameters is a regular sequence.*

(3) *For some homogeneous system of parameters $F_1, \ldots, F_n$, R is a free-module over $K[F_1, \ldots, F_n]$.*

(4) *For every homogeneous system of parameters $F_1, \ldots, F_n$, R is a free-module over $K[F_1, \ldots, F_n]$.*

(5) *R_m is Cohen-Macaulay.*

(6) *R is Cohen-Macaulay.*

Proof. The proof of the equivalence of (1) and (2) is the same as for the local case, already given above.

The preceding Lemma yields the equivalence of (1) and (3), as well as the equivalence of (2) and (4). Thus, (1) through (4) are equivalent.

It is clear that $(6) \Rightarrow (5)$. To see that $(5) \Rightarrow (2)$ consider a homogeneous system of parameters in R. It generates an ideal whose radical is m, and so it is also a system of parameters for R_m. Thus, the sequence is a regular sequence in R_m. We claim that it is also a regular sequence in R. If not, x_{k+1} is contained in an associated prime of $(x_1, \ldots, x_k)$ for some k, $0 \leq k \leq n-1$. Since the associated primes of a homogeneous ideal are homogeneous, this situation is preserved when we localize at m, which gives a contradiction.

To complete the proof, it will suffice to show that $(1) \Rightarrow (6)$. Let $F_1, \ldots, F_n$ be a homogeneous system of parameters for R. Then R is a free module over $A = K[F_1, \ldots, F_n]$, a polynomial ring. Let Q be any maximal ideal of R and let P denote its contraction to A, which will be maximal. These both have height n. Then $A_P \to R_Q$ is faithfully flat. Since A is regular, A_P is Cohen-Macaulay. Choose a system of parameters for A_P. These form a regular sequence in A_P, and, hence, in the faithfully flat extension R_Q. It follows that R_Q is Cohen-Macaulay. $\square$

From part (2) of the Lemma on p. 8 we also have:

Theorem. *Let R be a module-finite local extension of a regular local ring A. Then R is Cohen-Macaulay if and only if R is A-free.*

It it is not always the case that a local ring (R, m, K) is module-finite over a regular local ring in this way. But it does happen frequently in the complete case. Notice that the

property of being a regular sequence is preserved by completion, since the completion $\widehat{R}$ of a local ring is faithfully flat over R, and so is the property of being a system of parameters. Hence, R is Cohen-Macaulay if and only if $\widehat{R}$ is Cohen-Macaulay.

If R is complete and contains a field, then there is a coefficient field for R, i.e., a field $K \subseteq R$ that maps isomorphically onto the residue class field K of R. Then, if $x_1, \ldots, x_n$ is a system of parameters, R turns out to be module-finite over the formal power series ring $K[[x_1, \ldots, x_n]]$ in a natural way. Thus, in the complete equicharacteristic local case, we can always find a regular ring $A \subseteq R$ such that R is module-finite over A, and think of the Cohen-Macaulay property as in the Theorem above.

The structure theory of complete local rings is discussed in detail in the Lecture Notes from Math 615, Winter 2007: see the Lectures of March 21, 23, 26, 28, and 30 as well as the Lectures of April 2 and April 4.

Cohen-Macaulay modules

All of what we have said about Cohen-Macaulay rings generalizes to a theory of Cohen-Macaulay modules. We give a few of the basic definitions and results here: the proofs are very similar to the ring case, and are left to the reader.

If M is a module over a ring R, the *Krull dimension* of M is the Krull dimension of $R/\mathrm{Ann}_R(I)$. If (R, m, K) is local and $M \neq 0$ is finitely generated of Krull dimension d, a *system of parameters* for M is a sequence of elements $x_1, \ldots, x_d \in m$ such that, equivalently:

(1) $\dim\big(M/(x_1, \ldots, x_d)M\big) = 0$.

(2) The images of $x_1, \ldots, x_d$ form a system of parameters in $R/\mathrm{Ann}_R M$.

In this local situation, M is *Cohen-Macaulay* if one (equivalently, every) system of parameters for M is a regular sequence on M. If J is an ideal of $R/\mathrm{Ann}_R M$ of height h, then it contains part of a system of parameters for $R/\mathrm{Ann}_R M$ of height h, and this will be a regular sequence on M. It follows that the Cohen-Macaulay property for M passes to M_P for every prime P in the support of M. The arguments are all essentially the same as in the ring case.

If R is any Noetherian ring $M \neq 0$ is any finitely generated R-module, M is called *Cohen-Macaulay* if all of its localizations at maximal (equivalently, at prime) ideals in its support are Cohen-Macaulay.

The Cohen-Macaulay condition is increasingly restrictive as the Krull dimension increases. In dimension 0, every local ring is Cohen-Macaulay. In dimension one, it is sufficient, but not necessary, that the ring be reduced: the precise characterization in dimension one is that the maximal ideal not be an embbeded prime ideal of (0). Note that

$K[[x,y]]/(x^2)$ *is* Cohen-Macaulay, while $K[[x,y]]/(x^2, xy)$ is not. Also observe that all one-dimensional domains are Cohen-Macaulay.

In dimension 2, it suffices, but is not necessary, that the ring R be normal, i.e., integrally closed in its ring of fractions. Note that a normal Noetherian ring is a finite product of normal domains. If (R, m, K) is local and normal, then it is a doman. The associated primes of a principal ideal are minimal if R is normal. Hence, if x, y is a system of parameters, y is not in any associated prime of xR, i.e., it is not in any associated prime of the module R/xR, and so y is not a zerodivisor modulo xR.

The two dimensional domains $K[[x^2, x^2, y, xy]]$ and $K[x^4, x^3y, xy^3, y^4]]$ (one may also use single brackets) are not Cohen-Macaulay: as an exercise, the reader may try to see that y is a zerodivisor mod x^2 in the first, and that y^4 is a zerodivisor mod x^4 in the second. On the other hand, while $K[[x^2, x^3, y^2, y^3]]$ is not normal, it is Cohen-Macaulay.

Direct summands of rings

Let $R \subseteq S$ be rings. We want to discuss the consequences of the hypothesis that the inclusion $R \hookrightarrow S$ splits as a map of R-modules. When this occurs, we shall simply say that R is a *direct summand* of S. When we have such a splitting, we have an R-linear map $\rho : S \to R$ that is the identity on R. Here are some facts.

Proposition. *Let R be a direct summand of S. Then:*

(a) *For every ideal I of R, $IS \cap R = I$.*

(b) *If S is Noetherian, then R is Noetherian.*

(c) *If R is an $\mathbb{N}$-graded ring with $R_0 = A$ and S is Noetherian , then R is finitely generated over A.*

(d) *If S is a normal domain, then R is normal.*

Proof. Let ρ be a splitting.

(a) If $r \in R$ is such that $r = \sum_{i=1}^{h} f_i s_i$ with the $f_i \in I$ and the $s_i \in S$, so that r is a typical element of $IS \cap R$, then $r = \rho(r) = \sum_{i=1}^{n} f_i \rho(s_i)$, since the $f_i \in R$. Since each $\rho(s_i) \in R$, we have that $r \in I$.

(b) If $\{I_n\}_n$ is a nondecreasing chain of ideals of R, we have that the chain $\{I_n S\}_n$ is stable from some point on, say $I_t S = I_N S$ for all $t \geq N$. We may then apply (a) to obtain that $I_t = I_t S \cap R = I_N S \cap R = I_N$ for all $t \geq N$.

(c) From part (b), R is Noetherian, and so the ideal J spanned by all forms of positive degree is finitely generated, say by forms $F_1, \dots, F_n$ of positive degree. Then $R = A[F_1, \dots, F_n]$: otherwise, choose a form G of least degree that is in R and not in $A[F_1, \dots, F_n]$. Then $G \in J$, and so we can write G as a sum of terms $H_j F_j$ where every

H_j is a nonzero form such that $\deg(H_j) + \deg(F_j) = \deg(G)$. Since $\deg(H_j) < \deg(G)$, every $H_j \in A[F_1, \ldots, F_n]$, and the result follows.

(d) Let $a, b \in R$ with $b \neq 0$ such that a/b is integral over R. Then a/b is an element of $\operatorname{frac}(S)$ integral over S as well, and so $a/b \in S$. Thus, $a \in bS \cap R = bR$ by part (a). and so $a = br$ with $r \in R$. This shows that $r = a/b \in R$. $\square$

Segre products

Let R and S be finitely generated $\mathbb{N}$-graded K-algebras with $R_0 = S_0 = K$. We define the *Segre product* $R \circledS_K S$ of R and S over K to be the ring

$$\bigoplus_{n=1}^{\infty} R_n \otimes_K S_n,$$

which is a subring of $R \otimes_K S$. In fact, $R \otimes_K S$ has a grading by $\mathbb{N} \times \mathbb{N}$ whose (m, n) component is $R_m \otimes_K S_n$. (There is no completely standard notation for Segre products: the one used here is only one possbility.) The vector space

$$\bigoplus_{m \neq n} R_m \otimes_K S_n \subseteq R \otimes_K S$$

is an $R \circledS_K S$-submodule of $R \otimes_K S$ that is an $R \circledS_K S$-module complement for $R \circledS_K S$. That is, $R \circledS_K S$ is a direct summand of $R \otimes_K S$ when the latter is regarded as an $R \circledS_K S$-module. It follows that $R \circledS_K S$ is Noetherian and, hence, finitely generated over K. Moreover, if $R \otimes_K S$ is normal then so is $R \circledS_K S$. In particular, if R is normal and S is a polynomial ring over K then $R \circledS_K S$ is normal.

Let $S = K[X, Y, Z]/(X^3 + Y^3 + Z^3) = K[x, y, z]$, where K is a field of characteristic different from 3: this is a homogeneous coordinate ring of an elliptic curve C, and is often referred to as a *cubical cone*. Let $T = K[s, t]$, a polynomial ring, which is a homogeneous coordinate ring for the projective line $\mathbb{P}^1 = \mathbb{P}^1_K$. The Segre product of these two rings is $R = K[xs, ys, zs, xt, yt, zt] \subseteq S[s, t]$, which is a homogeneous coordinate ring for the smooth projective variety $C \times \mathbb{P}^1$. This ring is a normal domain with an isolated singularity at the origin: that is, its localization at any prime ideal except the homogeneous maximal ideal m is regular. R and R_m are normal but not Cohen-Macaulay.

We give a proof that R is not Cohen-Macaulay. The equations

$$(zs)^3 + \big((xs)^3 + (ys)^3\big) = 0 \quad \text{and} \quad (zt)^3 + \big((xt)^3 + (yt)^3\big) = 0$$

show that zs and zt are both integral over $D = K[xs, ys, xt, zt] \subseteq R$. The elements x, y, s, and t are algebraically independent, and the fraction field of D is $K(xs, ys, t/s)$, so that $\dim(D) = 3$, and

$$D \cong K[X_{11}, X_{12}, X_{21}, X_{22}]/(X_{11}X_{22} - X_{12}X_{21})$$

with $X_{11}, X_{12}, X_{21}, X_{22}$ mapping to xs, ys, xt, yt respectively.

It is then easy to see that ys, xt, $xs - yt$ is a homogeneous system of parameters for D, and, consequently, for R as well. The relation

$$(zs)(zt)(xs - yt) = (zs)^2(xt) - (zt)^2(ys)$$

now shows that R is *not* Cohen-Macaulay, for $(zs)(zt) \notin (xt, ys)R$. To see this, suppose otherwise. The map

$$K[x, y, z, s, t] \to K[x, y, z]$$

that fixes $K[x, y, z]$ while sending $s \mapsto 1$ and $t \mapsto 1$ restricts to give a K-algebra map

$$K[xs, ys, zs, xt, yt, zt] \to K[x, y, z].$$

If $(zs)(zt) \in (xt, ys)R$, applying this map gives $z^2 \in (x, y)K[x, y, z]$, which is false — in fact, $K[x, y, z]/(x, y) \cong K[z]/(z^3)$. $\quad\square$

Cohen-Macaulay rings are wonderfully well-behaved in many ways: we shall discuss this at considerable length later. Of course, regular rings are even better.

One of the main objectives in these lectures is to discuss two ways of dealing with rings in which the Cohen-Macaulay property fails. One is the development of a tight closure theory. The other is to prove the existence of "lots" of big Cohen-Macaulay algebras. These two methods are closely related, and we shall explore that relationship. In any case, one conclusion that one may reach is that rings that do not have the Cohen-Macaulay property nonetheless have better behavior than one might at first expect.

The situation right now is that there are relatively satisfactory results for both of these techniques for Noetherian rings containing a field. There are also results for local rings of mixed characteristic in dimension at most 3. (For a mixed characteristic local domain, the characteristic of the residue class field is a positive prime p while the characteristic of the fraction field is 0. The p-adic integers give an example, as well as module-finite extensions of formal power series rings over the p-adic integers.)

The integral closure of an ideal

The Briançon-Skoda theorem discussed in (2) below refers to the *integral closure $\bar{I}$* of an ideal I. We make the following comments: for proofs, see the Lecture Notes from Math 711, Fall 2006, September 13 and September 15 (those notes also give a detailed treatment of the Lipman-Sathaye proof of the Briançon-Skoda theorem). If $I \subseteq R$ and $u \in R$ then $u \in \bar{I}$ precisely if for some n, u satisfies a monic polynomial

$$x^n + r_1 x^{n-1} + \cdots + r_n = 0$$

with $r_j \in I^j$, $1 \le j \le n$.

Alternatively, if one forms the *Rees ring*

$$R[It] = R + It + I^2 t^2 + I^3 t^3 + \cdots + I^n t^n + \cdots \subseteq R[t],$$

where t is an indeterminate, the integral closure of $R[It]$ in $R[t]$ has the form

$$R + J_1 t + J_2 t^2 + J_3 t^3 + \cdots + J_n t^n + \cdots$$

where every $J_n \subseteq R$ is an ideal. It turns out that $J_1 = \bar{I}$, and, in fact, $J_n = \overline{I^n}$ for all $n \ge 1$.

It turns out as well that for $u \in R$, one has that $u \in \bar{I}$ if and only if $u \in IV$ for every map from R to a valuation domain V. When R is Noetherian, it suffices to consider maps to Noetherian discrete valuation domains (we refer to such a domain as a DVR: this is the same as a regular local ring of Krull dimension 1) such that the kernel of the map is a minimal prime of R. In particular, if R is a Noetherian domain, it suffices to consider injective maps of R into a DVR.

If R is a Noetherian domain, yet another characterization of $\bar{I}$ is as follows: $u \in \bar{I}$ if and only if there is an element $c \in R - \{0\}$ such that $cu^n \in I^n$ for all $n \in \mathbb{N}$ (it suffices if $cu^n \in I^n$ for infinitely many values of $n \in \mathbb{N}$).

Here are some of the results that can be proved using tight closure theory, which we shall present even though we have not yet discussed what tight closure is.

(1) If $R \subseteq T$ are rings such that T is regular and R is a direct summand of T as an R-module, then R is Cohen-Macaulay. (This is known in the equal characteristic case: it is an open question in general.)

(2) If $I = (f_1, \dots, f_n)$ is an ideal of a regular ring R, then $\overline{I^n} \subseteq I$. (The case where R is regular is known even in mixed characteristic. In the case where R is equicharacteristic, it is known that $\overline{I^n}$ is contained in the tight closure of I, with no restriction on the Noetherian ring R.)

(3) If $I \subseteq R$ is an ideal and S is module-finite extension of R, then $IS \cap R$ is contained in the tight closure of I in equal characteristic. (That is, tight closure "controls" how large the contracted expansion of an ideal to a module-finite extension ring can be.)

(4) Tight closure can be used to prove that if R is regular, then R is a direct summand of every module-finite extension ring. More generally, in equal characteristic, every ring such that every ideal is tightly closed is a direct summand of every module-finite extension ring. Whether the converse holds is an open question.

Whether every regular ring is a direct summand of every module-finite extension remains an important open question in mixed characteristic, where it is known in dimension at most 3. The proof in dimension 3, due to Ray Heitmann, is very difficult. We shall discuss Heitmann's work further.

(5) Tight closure can be used to prove theorems controlling the behavior of symbolic powers of prime ideals in regular rings. (We shall give more details about this in the next lecture.)

(6) Tight closure can be used in the proof of several subtle statements about homological properties of local rings. These statements are known as "the local homological conjectures." Some are now theorems in equal characteristc but open in mixed characteristic. Others are now known in general. Some remain open in every characteristic. We shall discuss these in more detail later.

By a *big Cohen-Macaulay* module for a local ring (R, m, K) we mean a not necessarily finitely generated R-module M such that every system of parameters of R is a regular sequence on M. It is not sufficient for one system of parameters to be a regular sequence, but if one system of parameters is a regular sequence then the m-adic completion of M has the property that every system of parameters is a regular sequence. Some authors use the term "big Cohen-Macaulay module" when one system of parameters is a regular sequence, and call the big Cohen-Macaulay module "balanced" if every system of parameters is a regular sequence.

An R-algebra S is called a *big Cohen-Macaulay* algebra over R if it is a big Cohen-Macaulay module as well as an R-algebra.

The existence of big Cohen-Macaulay algebras is known if the local ring R contains a field. The proof in equal characteristic 0 depends on reduction to characteristic $p > 0$. In mixed characteristic, it is easy in dimension at most 2 and follows from difficult results of Heitmann in dimension 3. We shall discuss all this at considerable length later.

Big Cohen-Macaulay algebras can be used to prove results like those mentioned in (1), (4), and (6) for tight closure. I conjecture that the existence of a tight closure theory with sufficiently good properties in mixed characteristic is equivalent to the existence of sufficiently many big Cohen-Macaulay algebras in mixed characteristic. This is a somewhat vague statement, in that I am not being precise about the meaning of the word "sufficiently" in either half, but it is a point of view that forms one of the themes of these lectures, and will be developed further.