Symbolic powers

We want to make a number of comments about the behavior of symbolic powers of prime ideals in Noetherian rings, and to give at least one example of the kind of theorem one can prove about symbolic powers of primes in regular rings: there was a reference to such theorems in (5) on p. 15 of the notes from the first lecture.

Let P be a prime ideal in any ring. We define the n th *symbolic power* $P^{(n)}$ of P as

$$\{r \in R : \text{for some } s \in R - P, sr \in P^n\}.$$

Alternatively, we may define $P^{(n)}$ as the contraction of $P^n R_P$ to R. It is the smallest P-primary ideal containing P^n. If R is Noetherian, it may be described as the P-primary component of P^n in its primary decomposition.

While $P^{(1)} = P$, and $P^{(n)} = P^n$ when P is a *maximal* ideal, in general $P^{(n)}$ is larger than P^n, even when the ring is regular. Here is one example. Let x, y, z, and t denote indeterminates over a field K. Grade $R = K[x, y, z]$ so that x, y, and z have degrees 3, 4, and 5, respectively. Then there is a degree preserving K-algebra surjection

$$R \twoheadrightarrow K[t^3, t^4, t^5] \subseteq K[t]$$

that sends x, y, and z to t^3, t^4, and t^5, respectively. Note that the matrix

$$X = \begin{pmatrix} x & y & z \\ y & z & x^2 \end{pmatrix}$$

is sent to the matrix

$$\begin{pmatrix} t^3 & t^4 & t^5 \\ t^4 & t^5 & t^6 \end{pmatrix}.$$

The second matrix has rank 1, and so the 2×2 minors of X are contained in the kernel P of the surjection $R \twoheadrightarrow K[t^3, t^4, t^5]$. Call these minors $f = xz - y^2$, $g = x^3 - yz$, and $h = yx^2 - z^2$. It is not difficult to prove that these three minors generate P, i.e., $P = (f, g, h)$. We shall exhibit an element of $P^{(2)} - P$. Note that f, g, and h are homogeneous of degrees 8, 9, and 10, respectively.

Next observe that $g^2 - fh$ vanishes mod xR: it becomes $(-yz)^2 - (-y^2)(-z^2) = 0$. Therefore, $g^2 - fh = xu$. g^2 has an x^6 term which is not canceled by any term in fh, so that $u \neq 0$. (Of course, we could check this by writing out what u is in a completely explicit calculation.) The element $g^2 - fh \in P^2$ is homogeneous of degree 18 and x has degree 3. Therefore, u has degree 15. Since $x \notin P$ and $xu \in P^2$, we have that $u \in P^{(2)}$.

1

But since the generators of P all have degree at least 8, the generators of P^2 all have degree at least 16. Since $\deg(u) = 15$, we have that $u \notin P^2$, as required.

Understanding symbolic powers is difficult. For example, it is true that if $P \subseteq Q$ are primes of a regular ring then $P^{(n)} \subseteq Q^{(n)}$: but this is somewhat difficult to prove! See the Lectures of October 20 and November 1, 6, and 8 of the Lecture Notes from Math 711, Fall, 2006.

This statement about inclusions fails in simple examples where the ring is not regular. For example, consider the ring

$$R = K[U,\ V,\ W,\ X,\ Y,\ Z]/(UX + VY + WZ) = K[u,\ v,\ w,\ x,\ y,\ z]$$

where the numerator is a polynomial ring. Then R is a hypersurface: it is Cohen-Macaulay, normal, with an isolated singularity. It can even be shown to be a UFD. Let Q be the maximal ideal generated by the images of all of the variables, and let P be the prime ideal $(v,\ w,\ x,\ y,\ z)R$. Here, $R/P \cong K[U]$. Then $P \subseteq Q$ but it is not true that $P^{(2)} \subseteq Q^{(2)}$. In fact, since $-ux = yy + wz \in P^2$ and $u \notin P$, we have that $x \in P^{(2)}$, while $x \notin Q^{(2)}$, which is simply Q^2 since Q is maximal.

The following example, due to Rees, shows that behavior of symbolic powers can be quite bad, even in low dimension.

Let P be a prime ideal in a Noetherian ring R. Let t be an indeterminate over R. When I is an ideal of R, a very standard construction is to form the Rees ring

$$R[It] = R + It + \cdots + I^n t^n + \cdots \subseteq R[t],$$

which is finitely generated over R: if $f_1,\ \ldots,\ f_h$ generate the ideal I, then

$$R[It] = R[f_1 t,\ \ldots,\ f_h t].$$

An analogous construction when $I = P$ is prime is the *symbolic power algebra*

$$R + Pt + P^{(2)} t^2 + \cdots + P^{(n)} t^n + \cdots \subseteq R[t].$$

We already know that this algebra is larger than $R[Pt]$, but one might still hope that it is finitely generated. Roughly speaking, this would say that the elements in $P^{(n)} - P^n$ for sufficiently large n arise a consequence of elements in $P^{(k)} - P^k$ for finitely many values of k.

However, this is false. Let

$$R = \mathbb{C}[X,\ Y,\ Z]/(X^3 + Y^3 + Z^3),$$

where $\mathbb{C}$ is the field of complex numbers. This is a two-dimensional normal surface: it has an isolated singularity. It is known that there are height one homogeneous primes P that have infinite order in the divisor class group: this simply means that no symbolic power of P is principal. David Rees proved that the symbolic power algebra of such a prime P is not finitely generated over R. This was one of the early indications that Hilbert's Fourteenth Problem might have a negative solution, i.e., that the ring of invariants of a linear action of a group of invertible matrices on a polynomial ring over a field K may have a ring of invariants that is not finitely generated over K. M. Nagata gave examples to show that this can happen in 1958.

$\vphantom{x}$

Analytic spread

In order to give a proof of the result of Rees described above, we introduce the notion of analytic spread. Let (R, m, K) be local and $I \subseteq m$ an ideal. When K is infinite, the following two integers coincide:

(1) The least integer n such that I is integral over an ideal $J \subseteq I$ that is generated by n elements.

(2) The Krull dimension of the ring $K \otimes_R R[It]$.

The integer defined in (2) is called the *analytic spread* of I, and we shall denote it $\mathfrak{an}(I)$. See the Lecture Notes of September 15 and 18 from Math 711, Fall 2006 for a more detailed treatment.

The ring in (2) may be written as

$$S = K \oplus I/mI \oplus I^2/mI^2 \oplus \cdots \oplus I^n/mI^n \oplus \cdots$$

Note that if we define the *associated graded ring* $\mathrm{gr}_I(R)$ of R with respect to I as

$$R \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots \oplus I^n/I^{n+1} \oplus \cdots,$$

which may also be thought of as $R[It]/IR[It]$, then it is also true that $S \cong K \otimes_R \mathrm{gr}_I(R)$.

The idea underlying the proof that when K is infinite and $h = \mathfrak{an}(I)$, one can find $f_1, \ldots, f_h \in I$ such that I is integral over $J = (f_1, \ldots, f_h)R$ is as follows. The K-algebras S is generated by its one-forms. If K is infinite, one can choose a homogeneous system of parameters for S consisting of one-forms: these are elements of I/mI, and are represented by elements $f_1, \ldots, f_h$ of I. Let J be the ideal generated by $f_1, \ldots, f_h$ in R. The S is module-finite over the image of $K \otimes R[Jt]$, and using this fact and Nakayama's Lemma on each component, one can show that $R[It]$ is integral over $R[Jt]$, from which it follows that I is integral over J.

Proof that Rees's symbolic power algebra is not finitely generated

Here is a sketch of Rees's argument. Assume that the symbolic power algebra is finitely generated. We now replace the graded ring R by its localization at the homogeneous maximal ideal. By the local and homogeneous versions of Nakayama's Lemma, the least number of generators of an ideal generated by homogeneous elements of positive degree does not change. It follows that P continues to have the property that no symbolic power is principal. We shall prove that the symbolic power algebra cannot be finitely generated even in this localized situation, which implies the result over the original ring R.

Henceforth, (R, m, K) is a normal local domain of dimension 2 and P is a height one prime such that no symbolic power of P is prinicpal. We shall show that the symbolic power algebra of P cannot be finitely generated over R. Assume that it is finitely generated.

This implies that for some integer k, $P^{(nk)} = (P^{(k)})^n$ for all positive integers n. Let $I = P^{(k)}$. The ring $S = R[It]$ has dimension 3, since the transcendence degree over R is one. The elements x, y are a system of parameters for R. We claim that there is a regular sequence of length two in m on each symbolic power $J = P^{(h)}$. To see this, we take x to be the first term. Consider J/xJ. If there is no choice for the second term, then the maximal ideal m of R must be an associated prime of J/xJ, and we can choose $v \in J - xJ$ such that $mv \subseteq xJ$. But then $yv \in xR$, and x, y is a regular sequence in R. It follows that $v = xu$ with $u \in R - J$. Then $mxu \subseteq xJ$ shows $mu \subseteq J$. But elements of $m - P$ are not zerodivisors on J, so that $u \in J$, a contradiction. It follows that every system of parameters in R is a regular sequence on J: J is a Cohen-Macaulay module.

Thus, if the symbolic power algebra is finitely generated, x, y is a regular sequence on every $P^{(n)}$, and therefore x, y is a regular sequence in S. It follows that killing (x, y) decreases the dimension of the ring S by two. Since the radical of (x, y) is the homogeneous maximal ideal of R, we see that $(R/m) \otimes_R S$ has dimension one. This shows that the analytic spread of I is one. But then I is integral over a principal ideal. In a normal ring, principal ideals are integrally closed. Thus, I is principal. But this contradicts the fact that no symbolic power of P is principal. $\square$

The notion of tight closure for ideals

We next want to introduce tight closure for ideals in prime characteristic $p > 0$. We need some notations. If R is a Noetherian ring, we use R° to denote the set of elements in R that are not in any minimal prime of R. If R is a domain, $R^\circ = R - \{0\}$. Of course, R° is a multiplicative system.

We shall use e to denote an element of $\mathbb{N}$, the nonnegative integers. For typographical convenience, shall use q as a symbol interchangeable with p^e, so that whenever one writes

q it is understood that there is a corresponding value of e such that $q = p^e$, even though it may be that e is not shown explicitly.

When R is an arbitrary ring of characteristic $p > 0$, we write F_R or simply F for the Frobenius endomorphism of the ring R. Thus, $F(r) = r^p$ for all $r \in R$. F_R^e or F^e indicates the e th iteration of F_R, so that $F^e(r) = r^q$ for all $r \in R$.

If R has characteristic p, $I^{[q]}$ denotes the ideal generated by all q th powers of elements of I. If one has generators for I, their q th powers generate $I^{[q]}$. (More generally, if $f : R \to S$ is any ring homomorphism and $I \subseteq R$ is an ideal with generators $\{r_\lambda\}_{\lambda \in \Lambda}$, the elements $\{f(r_\lambda)\}_{\lambda \in \Lambda}$ generate IS.)

Definition. Let R be a Noetherian ring of prime characteristic $p > 0$. Let $I \subseteq R$ be an ideal and let $u \in R$ be an element. Then u is in the *tight closure* of I in R, denoted I^*, if there exists $c \in R^\circ$ such that for all sufficiently large q, $cu^q \in I^{[q]}$.

This may seem like a very strange definition at first, but it turns out to be astonishingly useful. Of course, in presenting the definition, we might have written "for all sufficiently large e, $cu^{p^e} \in I^{[p^e]}$" instead.

The choice of c is allowed to depend on I and u, but *not* on q.

It is quite easy to see that I^* is an ideal containing I. Of great importance is the following fact, to be proved later:

Theorem. *Let R be a Noetherian ring of prime characteristic $p > 0$. If R is regular, then every ideal of R is tightly closed.*

If one were to use tight closure only to study regular rings, then one might think of this Theorem as asserting that the condition in the Definition above gives a criterion for when an element is in an ideal that, on the face of it, is somewhat weaker than being in the ideal. Even if the whole theory were limited in this fashion, it provides easy proofs of many results that cannot be readily obtained in any other way. We want to give a somewhat different way of thinking of the definition above. First note that it turns out that tight closure over a Noetherian ring can be tested modulo every minimal prime. Therefore, for many purposes, it suffices to consider the case of a domain.

Let R be any domain of prime characteristic $p > 0$. Within an algebraic closure L of the fraction field of R, we can form the ring $\{r^{1/q} : r \in R\}$. The Frobenius map F is an automorphism of L: this is the image of R under the inverse of F^e, and so is a subring of L isomorphic to R. We denote this ring $R^{1/q}$. This ring extension of R is unique up to canonical isomorphism: it is independent of the choice of L, and its only R-automorphism is the identity: r has a unique q th root in $R^{1/q}$, since the difference of two distinct q th roots would be nilpotent, and so every automorphism that fixes r fixes $r^{1/q}$ as well. Moreover,

there is a commutative diagram:

$$
\begin{array}{ccc}
R & \hookrightarrow & R^{1/q} \\
(*) \quad \| & & \uparrow{\scriptstyle F^{-e}} \\
R & \xrightarrow{\;F^e\;} & R
\end{array}
$$

where both vertical arrows are isomorphisms and $F^{-e}(r) = r^{1/q}$.

The reduced case

When R is reduced rather than a domain there is also a unique (up to unique isomorphism) reduced R-algebra extension ring $R^{1/q}$ whose elements are precisely all q th roots of elements of R. One can construct such an extension ring by taking the map $R\xrightarrow{F^e}R$ to give the algebra map, so that one has the same commutative diagram $(*)$ as in the domain case. The proof of uniqueness is straightforward: if S_1 and S_2 are two such extensions, the only possible isomorphism must let the unique q th root of $r \in R$ in S_1 correspond to the unique q th root of R in S_2 for all $r \in R$. It is easy to check that this gives a well-defined map that is the identity on R, and that it is a bijection and a homomorphism.

In both the domain and the reduced case, we have canonical embeddings $R^{1/q} \hookrightarrow R^{1/q'}$ when $q \leq q'$, and we define

$$
R^\infty = \bigcup_q R^{1/q}.
$$

When one has that

$$
cu^q = r_1 f_1^q + \cdots + r_h f_h^q
$$

one can take q th roots to obtain

$$
c^{1/q}u = r_1^{1/q} f_1 + \cdots + r_h^{1/q} f_h.
$$

Keep in mind that in a reduced ring, taking q th roots preserves the ring operations. We can therefore rephrase the definition of tight closure of an ideal I in a Noetherian domain R of characteristic $p > 0$ as follows:

(#) An element $u \in R$ is in I^* iff there is an element $c \in R^\circ$ such that for all sufficiently large q, $c^{1/q}u \in IR^{1/q}$.

Heuristically, one should think of an element of R that is in IS, where S is a domain that is an integral extension of R, as "almost" in I. Note that in this situation one will have $u = f_1 s_1 + \cdots + f_h s_h$ for $f_1, \ldots, f_h \in I$ and $s_1, \ldots, s_h \in S$, and so one also has $u \in IS_0$, where $S_0 = R[s_1, \ldots, s_h]$ is module-finite over R.

The condition (#) is weaker in a way: $R^{1/q} \subseteq R^\infty$ is an integral extension of R, but, u is not necessarily in IR^∞: instead, it is multiplied into IR^∞ by infinitely many elements $c^{1/q}$. These elements may be thought of as approaching 1 in some vague sense: this is not literally true for a topology, but the exponents $1/q \to 0$ as $q \to \infty$.

Some useful properties of tight closure

We state some properties of tight closure for ideals: proofs will be given later. Here, R is a Noetherian ring of prime characteristic $p > 0$, and I, J are ideals of R. We shall write R_{red} for the homomorphic image of R obtained by killing the ideal of nilpotent elements. (R, m, K) is called *equidimensional* if for every minimal prime P of R, $\dim(R/P) = \dim(R)$. An algebra over R is called *essentially of finite type* over R if it is a localization at some multiplicative system of a finitely generated R-algebra. If I, J are ideals of R, we define $I :_R J = \{r \in R : rJ \subseteq I\}$, which is an ideal of R. If $J = uR$, we may write $I :_R u$ for $I :_R uR$.

Excellent rings

In some of the statements below, we have used the term "excellent ring." The excellent rings form a subclass of Noetherian rings with many of the good properties of finitely generated algebras over fields and their localizations. We shall not give a full treatment in these notes, but we do discuss certain basic facts that we need. For the moment, the reader should know that the excellent rings include any ring that is a localization of a finitely generated algebra over a complete local (or semilocal) ring. The class is closed under localization at any multiplicative system, under taking homomorphic images, and under formation of finitely generated algebras. We give more detail later. Typically, Noetherian rings arising in algebraic geometry, number theory, and several complex variables are excellent.

Here are nine properties of tight closure. Property (2) was already stated as a Theorem earlier.

(1) $I \subseteq I^* = (I^*)^*$. If $I \subseteq J$, then $I^* \subseteq J^*$.

(2) If R is regular, every ideal of R is tightly closed.

(3) If $R \subseteq S$ is a module-finite extension, $IS \cap R \subseteq I^*$.

(4) If $P_1, \ldots, P_h$ are the minimal primes of R, then $u \in R$ is in I^* if and only if the image of u in $D_j = R/P_j$ is in the tight closure of ID_j in D_j, working over D_j, for $1 \le j \le h$.

(5) If $u \in R$ then $u \in I^*$ if and only if its image in R_{red} is in the tight closure of IR_{red}, working over R_{red}.

The statements in (4) and (5) show that the study of tight closure can often be reduced to the case where R is reduced or even a domain.

The following is one of the most important properties of tight closure. It is what enables one to use tight closure as a substitute for the Cohen-Macaulay property in many instances. It is the key to proving that direct summands of regular rings are Cohen-Macaulay in characteristic $p > 0$.

(6) **(Colon-capturing)** If $(R,\, m,\, K)$ is a complete local domain (more generally, if $(R,\, m,\, K)$ is a reduced, excellent, and equidimensional), the elements $x_1,\, \ldots,\, x_k,\, x_{k+1}$ are part of a system of parameters for R, and $I_k = (x_1,\, \ldots,\, x_k)R$, then $(x_1,\, \ldots,\, x_k) :_R x_{k+1} \subseteq I_k^*$.

Of course, if R were Cohen-Macaulay then we would have $I_k :_R x_{k+1} = I_k$.

(7) Under mild conditions on R, $u \in R$ is in the tight closure of $I \subseteq R$ if and only if the image of u in R_P is in the tight closure of $I R_P$, working over R_P, for all prime (respectively, maximal) ideals P of R. (The result holds, in particular, for algebras essentially of finite type over an excellent semilocal ring.)

Tight closure is not known to commute with localization, and this is now believed likely to be false. But property (7) shows that it has an important form of compatibility with localization.

(8) If $(R,\, m,\, K)$ is excellent, $I^* = \bigcap_n (I + m^n)^*$.

Property (8) shows that tight closure is determined by its behavior on m-primary ideals in the excellent case.

(9) If $(R,\, m,\, K)$ is reduced and excellent, $u \in I^*$ if and only if u is in the tight closure of $I\widehat{R}$ in $\widehat{R}$ working over $\widehat{R}$.

These properties together show that for a large class of rings, tight closure is determined by its behavior in complete local rings and, in fact, in complete local domains. Moreover, in a complete local domain it is determined by its behavior on m-primary ideals.

We next want to give several further characterizations of tight closure, although these require some additional condition on the ring. For the first of these, we need to discuss the notion of R^+ for a domain R first.

The absolute integral closure R^+ of a domain R

Let R be any integral domain (there are no finiteness restrictions, and no restriction on the characteristic). By an *absolute integral closure* of R, we mean the integral closure of R in an algebraic closure of its fraction field. It is immediate that R^+ is unique up to non-unique isomorphism, just as the algebraic closure of a field is.

Consider any domain extension S of R that is integral over R. Then the fraction field $\operatorname{frac}(R)$ is contained in the algebraic closure L of $\operatorname{frac}(S)$, and L is also an algebraic closure for R, since the elements of S are integral over R and, hence, algebraic over $\operatorname{frac}(R)$.

The algebraic closure of R in L is R^+. Thus, we have an embedding $S \hookrightarrow R^+$ as R-algebras. Therefore, R^+ is a maximal domain extension of R that is integral over R: this characterizes R^+. It is also clear that $(R^+)^+ = R^+$. When $R = R^+$ we say that R is *absolutely integrally closed*. The reader can easily verify that a domain S is absolutely integrally closed if and only if every monic polynomial in one variable $f \in S[x]$ factors into monic linear factors over S. It is easy to check that a localization at any multiplicative system of an absolutely integrally closed domain is absolutely integrally closed, and that a domain that is a homomorphic image of an absolutely integrally closed domain is absolutely integrally closed. (A monic polynomial over S/P lifts to a monic polynomial over S, whose factorization into monic linear factors gives such a factorizaton of the original polynomial over S/P.)

If $S \hookrightarrow T$ is an extension of domains, the algebraic closure of the fraction field of S contains an algebraic closure of the fraction field of R. Thus, we have a commutative diagram

$$
\begin{array}{ccc}
S^+ & \hookrightarrow & T^+ \\
\uparrow & & \uparrow \\
S & \hookrightarrow & T
\end{array}
$$

where the vertical maps are inclusions.

If $R \twoheadrightarrow S$ is a surjection of domains, so that $S \cong R/P$, by the lying-over theorem there is a prime ideal Q of R^+ lying over P, since $R \hookrightarrow R^+$ is an integral extension. Then $R \to R^+/Q$ has kernel $Q \cap R = P$, and so we have $S \cong R/P \hookrightarrow R^+/Q$. Since R^+ is integral over R, R^+/Q is integral over $R/P \cong S$. But since R^+ is absolutely integrally closed, so is R^+/Q. Thus, R^+/Q is an integral extenson of S, and is an absolutely integrally closed domain. It follows that we may identify this extension with S^+, and so we have a commutative diagram

$$
\begin{array}{ccc}
R^+ & \twoheadrightarrow & S^+ \\
\uparrow & & \uparrow \\
R & \twoheadrightarrow & S
\end{array}
$$

where both vertical maps are inclusions.

Any homomorphism of domains $R \to T$ factors $R \twoheadrightarrow S \hookrightarrow T$ where S is the image of R in T. The two facts that we have proved yield a commutative diagram

$$
\begin{array}{ccccc}
R^+ & \twoheadrightarrow & S^+ & \hookrightarrow & T^+ \\
\uparrow & & \uparrow & & \uparrow \\
R & \twoheadrightarrow & S & \hookrightarrow & T
\end{array}
$$

where all of the vertical maps are inclusions. Hence:

Proposition. *For any homomorphism $R \to T$ of integral domains there is a commutative diagram*

$$
\begin{array}{ccc}
R^+ & \longrightarrow & T^+ \\
\uparrow & & \uparrow \\
R & \longrightarrow & T
\end{array}
$$

where both vertical maps are inclusions. □

Other characterizations of tight closure

For many purposes it suffices to characterize tight closure in the case of a complete local domain. Let (R, m, K) be a complete local domain of prime characteristic $p > 0$. One can always choose a DVR $(V, t_V V, L)$ containing R such that $R \subseteq V$ is local. This gives a $\mathbb{Z}$-valued valuation nonnegative on R and positive on m. This valuation extends to a $\mathbb{Q}$-valued valuation on R^+. To see this, note that $R^+ \subseteq V^+$. V^+ is a directed union of module-finite normal local extensions W of V, each of which is a DVR. Let t_W be the generator of the maximal ideal of W. Then $t_V = t_W^{h_W} \alpha$ for some positive integer h_W and unit α of W, and we can extend the valuation to W by letting the order of t_W be $1/h_W$. (To construct V in the first place, we may write R as a module-finite extension of a complete regular local ring (A, m_A, K). By the remarks above, it suffices to construct the required DVR for A. There are many possibilities. One is to define the order of a nonzero element $a \in A$ to be the largest integer k such that $u \in m_A^k$. This gives a valuation because $\mathrm{gr}_{m_A} A$ is a polynomial ring over K, and, in particular, a domain.)

Theorem. *Let (R, m, K) be a complete local domain of prime characteristic $p > 0$, $u \in R$, and $I \subseteq R$. Choose a complete DVR (V, m_V, L) containing (R, m, K) such that $R \subseteq V$ is local. Extend the valuation on R given by V to a $\mathbb{Q}$-valued valuation on R^+: call this ord. Then $u \in I^*$ if and only if there exists a sequence of nonzero elements $c_n \in R^+$ such that for all n, $c_n u \in IR^+$ and $\mathrm{ord}(c_n) \to 0$ as $n \to \infty$.*

See Theorem (3.1) of [M. Hochster and C. Huneke, *Tight closure and elements of small order in integral extensions*, J. of Pure and Applied Alg. **71** (1991) 233–247].

This is clearly a necessary condition for u to be in the tight closure of I. We have $R^{1/q} \subseteq R^\infty \subseteq R^+$, and in so in the reformulation (#) of the definition of tight closure for the domain case, one has $c^{1/q} u \in IR^{1/q} \subseteq IR^+$ for all sufficiently large q. Since one has

$$
\mathrm{ord}\,(c^{1/q}) = \frac{1}{q}\,\mathrm{ord}\,(c),
$$

we may use the elements $c^{1/q}$ to form the required sequence. What is surprising in the theorem above is that one can use arbitrary, completely unrelated multipliers in testing for tight closure, and u is still forced to be in I^*.

Solid modules and algebras and solid closure

Ler R be any domain. An R-module M is called *solid* if it has a nonzero R-linear map $M \to R$. That is, $\mathrm{Hom}_R(M, R) \neq 0$.

An R-algebra S is called *solid* if it is solid as an R-module. In this case, we can actually find an R-linear map $\theta : S \to R$ such that $\theta(1) \neq 0$. For if θ_0 is any nonzero map $S \to M$, we can choose $s \in S$ such that $\theta_0(s) \neq 0$, and then define θ by $\theta(u) = \theta_0(su)$ for all $u \in S$.

When R is a Noetherian domain and M is a finitely generated R-module, the property of being solid is easy to understand. It simply means that M is not a torsion module over R. In this case, we can kill the torsion submodule N of M, and the torsion-free module M/N will embed a free module R^h. One of the coordinate projections π_j will be nonzero on M/N, and the composite

$$M \twoheadrightarrow M/N \hookrightarrow R^h \xrightarrow{\pi_j} R$$

will give the required nonzero map.

However, if S is a finitely generated R-algebra it is oftem very difficult to determine whether M is solid or not.

For those familiar with local cohomology, we note that if $(R,\, m,\, K)$ is a complete local domain of Krull dimension d, then M is solid over R if and only $H^d_m(M) \neq 0$. Local cohomology theory will be developed in supplementary lectures, and we will eventually prove this criterion. This criterion can be used to show the following.

Theorem. *Let $(R,\, m,\, K)$ be a complete local domain. Then a big Cohen-Macaulay algebra for R is solid.*

We will eventually prove the following characterization of tight closure for complete local domains. This result begins to show the close connection between tight closure and the existence of big Cohen-Macaulay algebras.

Theorem. *Let $(R,\, m,\, K)$ be a complete local domain of prime characteristic $p > 0$. Let $u \in R$. Let $I \subseteq R$ be an ideal. The following conditions are equivalent:*

(1) *$u \in I^*$.*

(2) *There exists a solid R-algebra S such that $u \in IS$.*

(3) *There exists a big Cohen-Macaulay algebra S over R such that $u \in IS$.*

Of course, (3) $\Rightarrow$ (2) is immediate from the preceding theorem. Conditions (2) and (3) are of considerable interest because they characterize tight closure without referrring to the Frobenius endomorphism, and thereby suggest closure operations not necessarily in characteristic $p > 0$ that may be useful. The characterization (2) leads to a notion of "solid closure" which has many properties of tight closure in dimension at most 2. In equal

characteristic 0 in dimension 3 and higher it appears to be the wrong notion, in that ideals of regular rings need not be closed. However, whether solid closure gives a really useful theory in mixed characteristic in dimension 3 and higher remains mysterious.

The characterization in (3) suggests defining a "big Cohen-Macaulay algebra" closure. This is promising idea in all characteristics and all dimensions, but the existence of big Cohen-Macaulay algebras in mixed characteristic and dimension 4 and higher remains unsettled. One of our goals is to explore what is understood about this problem.