Math 711: Lecture of September 12, 2007

In our treatment of tight closure for modules it will be convenient to use the Frobenius functors, which we view as special cases of base change. We first review some basic facts about base change.

Base change

If $f : R \to S$ is an ring homomorphism, there is a base change functor $S \otimes_R \_$ from R-modules to S-modules. It takes the R-module M to the R-module $S \otimes_R M$ and the map $h : M \to N$ to the unique S-linear map $S \otimes_R M \to S \otimes_R N$ that sends $s \otimes u \mapsto s \otimes h(u)$ for all $s \in S$ and $u \in M$. This map may be denoted $\mathbf{1}_S \otimes_R h$ or $S \otimes_R h$. Evidently, base change from R to S is a covariant functor. We shall temporarily denote this functor as $\mathcal{B}_{R \to S}$. It also has the following properties.

(1) Base change takes R to S.

(2) Base change commutes with arbitrary direct sums and with arbitrary direct limits.

(3) Base change takes R^n to S^n and free modules to free modules.

(4) Base change takes projective R-modules to projective S-modules.

(5) Base change takes flat R-modules to flat S-modules.

(6) Base change is right exact: if

$$M' \to M \to M'' \to 0$$

is exact, then so is

$$S \otimes_R M' \to S \otimes_R M \to S \otimes_R M'' \to 0.$$

(7) Base change takes finitely generated modules to finitely generated modules: the number of generators does not increase.

(8) Base change takes the cokernel of the matrix $\bigl(r_{ij}\bigr)$ to the cokernel of the matrix $\bigl(f(r_{ij})\bigr)$.

(9) Base change takes R/I to S/IS.

(10) For every R-module M there is a natural R-lineaar map $M \to S \otimes M$ that sends $u \mapsto 1 \otimes u$. More precisely, R-linearity means that $ru \mapsto g(r)(1 \otimes u) = g(r) \otimes u$ for all $r \in R$ and $u \in M$.

(11) Given homomorphisms $R \to S$ and $S \to T$, the base change functor $\mathcal{B}_{R \to T}$ for the composite homomorphism $R \to T$ is the composition $\mathcal{B}_{S \to T} \circ \mathcal{B}_{R \to S}$.

Part (1) is immediate from the definition. Part (2) holds because tensor product commutes with arbitrary direct sums and arbitrary direct limits. Part (3) is immediate from parts (1) and (2). If P is a projective R-module, one can choose Q such that $P \oplus Q$ is free. Then $(S \otimes_R P) \oplus (S \otimes_R Q)$ is free over S, and it follows that both direct summands are projective over S. Part (5) follows because if M is an R-module, the functor $(S \otimes_R M) \otimes_S \_$ on S-modules may be identified with the functor $M \otimes_R \_$ on S-modules. We have

$$(S \otimes_R M) \otimes_S U \cong (M \otimes_R S) \otimes_S U \cong M \otimes_R M,$$

by the associativity of tensor. Part (6) follows from the corresponding general fact for tensor products. Part (7) is immediate, for if M is finitely generated by n elements, we have a surjection $R^n \twoheadrightarrow M$, and this yields $S^n \twoheadrightarrow S \otimes_R M$. Part (8) is immediate from part (6), and part (9) is a consequence of (6) as well. (10) is completely straightforward, and (11) follows at once from the associativity of tensor products.

The Frobenius functors

Let R be a ring of prime characteristic $p > 0$. The *Frobenius* or *Peskine-Szpiro* functor $\mathcal{F}_R$ from R-modules to R-modules is simply the base change functor for $f : R \to S$ when $S = R$ and the homomorphism $f : R \to S$ is the Frobenius endomorphism $F : R \to R$, i.e, $F(r) = r^p$ for all $r \in R$. We may take the e-fold iterated composition of this functor with itself, which we denote $\mathcal{F}^e_R$. This is the same as the base change functor for the homomorphism $F^e : R \to R$, where $F^e(r) = r^{p^e}$ for all $r \in R$, by the iterated application of (11) above. When the ring is clear from context, the subscript ${}_R$ is omitted, and we simply write $\mathcal{F}$ or $\mathcal{F}^e$.

We then have, from the corresponding facts above:

(1) $\mathcal{F}^e(R) = R$.

(2) $\mathcal{F}^e$ commutes with arbitrary direct sums and with arbitrary direct limits.

(3) $\mathcal{F}^e(R^n) = R^n$ and $\mathcal{F}^e$ takes free modules to free modules.

(4) $\mathcal{F}^e$ takes projective R-modules to projective R-modules.

(5) $\mathcal{F}^e$ takes flat R-modules to flat R-modules.

(6) $\mathcal{F}^e$ is right exact: if

$$M' \to M \to M'' \to 0$$

is exact, then so is

$$\mathcal{F}^e(M') \to \mathcal{F}^e(M) \to \mathcal{F}^e(M'') \to 0.$$

(7) $\mathcal{F}^e$ takes finitely generated modules to finitely generated modules: the number of generators does not increase.

(8) $\mathcal{F}^e$ takes the cokernel of the matrix $\big(r_{ij}\big)$ to the cokernel of the matrix $\big(r_{ij}^{p^e}\big)$.

(9) $\mathcal{F}^e$ takes R/I to $R/I^{[q]}R$.

By part (10) in the list of properties of base change, for every R-module M there is a natural map $M \to \mathcal{F}^e(M)$. We shall use u^q to denote the image of u under this map, which agrees with usual the usual notation when $M = R$. R-linearity then takes the following form:

(10) For every R-module M the natural map $M \to \mathcal{F}^e(M)$ is such that for all $r \in R$ and all $u \in M$, $(ru)^q = r^q u^q$.

We also note the following: given a homomorphism $g : R \to S$ of rings of prime characteristic $p > 0$, we always have that $g \circ F_R^e = F_S^e \circ g$. In fact, all this says is that $g(r^q) = g(r)^q$ for all $r \in R$. This yields a corresponding isomorphism of compositions of base change functors:

(11) Let $R \to S$ be a homomorphism of rings of prime characteristic $p > 0$. Then for every R-module M, there is an identification $S \otimes_R \mathcal{F}_R^e(M) \cong \mathcal{F}_S^e(S \otimes_R M)$ that is natural in the R-module M.

When $N \subseteq M$ the map $\mathcal{F}^e(N) \to \mathcal{F}^e(M)$ need not be injective. We denote that image of this map by $N^{[q]}$ or, more precisely, by $N_M^{[q]}$. *However, one should keep in mind that* $N^{[q]}$ *is a submodule of* $\mathcal{F}^e(M)$, **not** *of* M *itself.* It is very easy to see that $N^{[q]}$ is the R-span of the elements of $\mathcal{F}^e(M)$ of the form u^q for $u \in N$. The module $N^{[q]}$ is also the R-span of the elements u_λ^q as u_λ runs through any set of generators for N.

A very important special case is when $M = R$ and $N = I$, an ideal of R. In this situation, $I_R^{[q]}$ is the same as $I^{[q]}$ as defined earlier. What happens here is atypical, because $F^e(R) = R$ for all e.

Tight closure for modules

Let R be a Noetherian ring of prime characteristic $p > 0$. If $N \subseteq M$, we define the *tight closure* N_M^* of N in M to consist of all elements $u \in M$ such that for some $c \in R^\circ$,

$$cu^q \in N_M^{[q]} \subseteq \mathcal{F}^e(M)$$

for all $q \gg 0$. Evidently, this agrees with our definition of tight closure for an ideal I, which is the case where $M = R$ and $N = I$. If M is clear from context, the subscript $_M$ is omitted, and we write N^* for N_M^*. Notice that we have not assumed that M or N is finitely generated. The theory of tight closure in Artinian modules is of very great interest. Note that c may depend on M, N, and even u. However, c is *not* permitted to depend on q. Here are some properties of tight closure:

Proposition. *Let R be a Noetherian ring of prime characteristic $p > 0$, and let N, M, and Q be R-modules.*

(a) N_M^* *is an R-module.*

(b) *If $N \subseteq M \subseteq Q$ are R-modules, then $N^*_Q \subseteq M^*_Q$ and $N^*_M \subseteq N^*_Q$.*

(c) *If $N_\lambda \subseteq M_\lambda$ is any family of inclusions, and $N = \bigoplus_\lambda N_\lambda \subseteq \bigoplus_\lambda M_\lambda = M$, then $N^*_M = \bigoplus_\lambda (N^*_\lambda)_{M_\lambda}$.*

(d) *If R is a finite product of rings $R_1 \times \cdots \times R_n$, $N_i \subseteq M_i$ are R_i-modules, $1 \leq i \leq n$, M is the R-module $M_1 \times \cdots \times M_n$, and $N \subseteq M$ is $N_1 \times \cdots \times N_n$, then N^*_M may be identify with $(N_1)^*_{M_1} \times \cdots \times (N_n)^*_{M_n}$.*

(e) *If I is an ideal of R, $I^* N^*_M \subseteq (IN)^*_M$.*

(f) *If $N \subseteq M$ and $V \subseteq W$ are R-modules and $h : M \to W$ is an R-linear map such that $h(N) \subseteq V$, then $h(N^*_M) \subseteq V^*_W$.*

Proof. (a) Let $c, c' \in R^\circ$. If $cu^q \in N^{[q]}$ for $q \geq q_0$, then $c(ru)^q \in N^{[q]}$ for $q \geq q_0$. If $c'v^q \in N^q$ for $q \geq q_1$ then $(cc')(u+v)^q \in N^{[q]}$ for $q \geq \max\{q_0, q_1\}$.

(b) The first statment holds because we have that $N^{[q]}_Q \subseteq M^{[q]}_Q$ for all q, and the second because the map $F^e(M) \to F^e(Q)$ carries $N^{[q]}_M$ into $N^{[q]}_Q$.

(c) is a straightforward application of the fact that tensor product commutes with direct sum and the definition of tight closure. Keep in mind that every element of the direct sum has nonzero components from only finitely many of the modules.

(d) is clear: note that $(R_1 \times \cdots \times R_n)^\circ = R_1^\circ \times \cdots \times R_n^\circ$.

(e) If $c, c' \in R^\circ$, $cf^q \in I^{[q]}$ for $q \gg 0$, and $c'u^{[q]} \in N^{[q]}$ for $q \gg 0$, then $(cc')(fu)^q = (cf^q)(c'u^q) \in I^{[q]}N^{[q]}$ for $q \gg 0$, and $I^{[q]}N^{[q]} = (IN)^{[q]}$ for every q.

(f) This argument is left as an exercise. $\square$

Let R and S be Noetherian rings of prime characteristic $p > 0$. We will frequently be in the situation where we want to study the effect of base change on tight closure. For this purpose, when $N \subseteq M$ are R-modules, it will be convenient to use the notation $\langle S \otimes_R N \rangle$ for the image of $S \otimes_R N$ in $S \otimes_R M$. Of course, one must know what the map $N \hookrightarrow M$ is, not just what N is, to be able to interpret this notation. Therefore, we may also use the more informative notation $\langle S \otimes_R N \rangle_M$ in cases where it is not clear what M is. Note that in the case where $M = R$ and $N = I \subseteq R$, $\langle S \otimes_R I \rangle = IS$, the expansion of I to S. More generally, if $N \subseteq G$, where G is free, we may write NS for $\langle S \otimes_R N \rangle_G \subseteq S \otimes G$, and refer to NS as the *expansion* of N, by analogy with the ideal case.

Proposition. *Let $R \to S$ be a homomorphism of Noetherian rings of prime characteristic $p > 0$ such that R° maps into S°. In particular, this hypothesis holds (1) if $R \subseteq S$ are domains, (2) if $R \to S$ is flat, or if (3) $S = R/P$ where P is a minimal prime of S. Then for all modules $N \subseteq M$, $\langle S \otimes_R N^*_M \rangle_M \subseteq (\langle S \otimes_R N \rangle_M)^*_{S \otimes_R M}$.*

Proof. It suffices to show that if $u \in N^*$ then $1 \otimes u \in \langle S \otimes_R N \rangle^*$. Since the image of c is in S°, this follows because $c(1 \otimes u^q) = 1 \otimes cu^q \in \langle S \otimes_R N^{[q]} \rangle = \langle S \otimes_R N \rangle^{[q]}$.

The statement about when the hypothesis holds is easily checked: the only case that is not immediate from the definition is when $R \to S$ is flat. This can be checked by proving that every minimal prime Q of S lies over a minimal prime P of R. But the induced map of localizations $R_P \to S_Q$ is faithfully flat, and so injective, and QS_Q is nilpotent, which shows that PR_P is nilpotent. $\square$

Tight closure, like integral closure, can be checked modulo every minimal prime of R.

Theorem. *Let R be a Noetherian ring of prime characteristic $p > 0$. Let $P_1, \ldots, P_n$ be the minimal primes of R. Let $D_i = R/P_i$. Let $N \subseteq M$ be R-modules, and let $u \in M$. Let $M_i = D_i \otimes_R M = M/P_iM$, and let $N_i = \langle D_i \otimes_R N \rangle$. Let u_i be the image of u in M_i. Then $u \in N^*_M$ over R if and only if for all i, $1 \le i \le n$, $u_i \in (N_i)^*_{M_i}$ over D_i.*

If $M = R$ and $N = I$, we have that $u \in I^*$ if and only if the image of u in D_i is in $(ID_i)^*$ in D_i, working over D_i, for all i, $1 \le i \le n$.

Proof. The final statement is just a special case of the Theorem. The "only if" part follows from the preceding Proposition. It remains to prove that if u is in the tight closure modulo every P_i, then it is in the tight closure. This means that for every i there exists $c_i \in R - P_i$ such that for all $q \gg 0$, $c_i u^q \in N^{[q]} + P_i F^e(M)$, since $\mathcal{F}^e(M/P_iM)$ working over D_i may be identified with $\mathcal{F}^e(M)/P_i\mathcal{F}^e(M)$. Choose d_i so that it is in all the P_j except P_i. Let J be the intersection of the P_i, which is the ideal of all nilpotents. Then for all i and all $q \gg 0$,

$$(*_i) \quad d_i c_i u^q \in N^{[q]} + JF^e(M),$$

since every $d_i P_i \subseteq J$.

Then $c = \sum_{i=1}^n d_i c_i$ cannot be contained in the union of P_i, since for all i the i th term in the sum is contained in all of the P_j except P_i. Adding the equations $(*_i)$ yields

$$cu^q \in N^{[q]} + JF^e(M)$$

for all $q \gg 0$, say for all $q \ge q_0$. Choose q_1 such that $J^{[q_1]} = 0$. Then $c^{q_1} u^{qq_1} \in N^{[qq_1]}$ for all $q \ge q_0$, which implies that $c^q u^q \in N^{[q]}$ for all $q \ge q_1 q_0$. $\square$

Let R have minimal primes $P_1, \ldots, P_n$, and let $J = P_1 \cap \cdots \cap P_n$, the ideal of nilpotent elements of R, so that $R_{\text{red}} = R/J$. The minimal primes of R/J are the ideals P_i/J, and for every i, $R_{\text{red}}/(P_i/J) \cong R/P_i$. Hence:

Corollary. *Let R be a Noetherian ring of prime characteristic $p > 0$, and let J be the ideal of all nilpotent elements of R. Let $N \subseteq M$ be R-modules, and let $u \in M$. Then $u \in N^*_M$ if and only if the image of u in M/JM is in $\langle N/J \rangle^*_{M/JM}$ working over $R_{\text{red}} = R/J$.*

We should point out that it is easy to prove the result of the Corollary directly without using the preceding Theorem.

We also note the following easy fact:

Proposition. *Let R be a Noetherian ring of prime characteristic $p > 0$. Let $N \subseteq M$ be R-modules. If $u \in N^*_M$, then for all $q_0 = p^{e_0}$, $u^{q_0} \in (N^{[q_0]})^*_{\mathcal{F}^{e_0}(M)}$.*

Proof. This is immediate from the fact that $(N^{[q_0]})^{[q]} \subseteq \mathcal{F}^e\big(\mathcal{F}^{e_0}(M)\big)$, if we identify the latter with $\mathcal{F}^{e_0+e}(M)$, is the same as $N^{[q_0q]}$. $\square$

We next want to consider what happens when we iterate the tight closure operation. When M is finitely generated, and quite a bit more generally, we do not get anything new. Later we shall develop a theory of *test elements* for tight closure that will enable us to prove corresponding results for a large class of rings without any finiteness conditions on the modules.

Theorem. *Let R be a Noetherian ring of prime characteristic $p > 0$, and let $N \subseteq M$ be R-modules. Consider the condtion :*

(#) *there exist an element $c \in R^\circ$ and $q_0 = p^{e_0}$ such that for all $u \in N^*$, $cu^q \in N^{[q]}$ for all $q \geq q_0$,*

which holds whenever N^*/N is a finitely generated R-module. If (#) holds, then $(N^*_M)^*_M = N^*_M$.

Proof. We first check that (#) holds when N^*/N is finitely generated. Let $u_1, \ldots, u_n$ be elements of N^* whose images generate N^*/N. Then for every i we can choose $c_i \in R^\circ$ and q_i such that for all $q \geq q_i$, we have that $c_i u^q \in N^{[q]}$ for all $q \geq q_i$. Let $c = c_1 \cdots c_n$ and let $q_0 = \max\{q_1, \ldots, q_n\}$. Then for all $q \geq q_0$, $cu_i^q \in N^{[q]}$, and if $u \in N$, the same condition obviously holds. Since every element of N^* has the form $r_1u_1 + \cdots + r_nu_n + u$ where the $r_i \in R$ and $u \in N$, it follows that (#) holds.

Now assume # and let $v \in (N^*)^*$. Then there exists $d \in R^\circ$ and q' such that for all $q \geq q'$, $dv^q \in (N^*)^{[q]}$, and so dv^q is in the span of elements w^q for $w \in N^*$. If $q \geq q_0$, we know that every $cw^q \in N^{[q]}$. Hence, for all $q \geq \max\{q', q_0\}$, we have that $(cd)v^q \in N^{[q]}$, and it follows that $v \in N^*$. $\square$

Of course, if M is Noetherian, then so is N^*, and condition (#) holds. Thus:

Corollary. *Let R be a Noetherian ring of prime characteristic $p > 0$, and let $N \subseteq M$ be finitely generated R-modules. Then $(N^*_M)^*_M = N^*_M$.* $\square$