**Math 711: Lecture of September 17, 2007**

**Definition.** Let $R$ be a Noetherian ring of prime characteristic $p > 0$. $R$ is called *weakly F-regular* if every ideal is tightly closed. $R$ is called *F-regular* if all of its localizations are weakly F-regular.

It is an open question whether, under mild conditions, e.g., excellence, weakly F-regular implies F-regular.

We shall show eventually that over a weakly F-regular ring, every submodule of every finitely generated module is tightly closed.

Since we have already proved that every regular ring of prime characteristic $p > 0$ is weakly F-regular and since the class of regular rings is closed under localization, it follows that every regular ring is F-regular.

We note the following fact.

**Lemma.** *Let $R$ be any Noetherian ring, let $M$ be a finitely generated module, and let $u \in M$. Suppose that $N \subseteq M$ is maximal with respect to the condition that $u \notin N$. Then $M/N$ has finite length, and it has a unique associated prime, which is a maximal ideal $m$ with a power that kills $M$. In this case $u$ spans the socle $\mathrm{Ann}_{MN} m$ of $M/N$.*

*Proof.* The maximality of $N$ implies that the image of $u$ is in every nonzero submodule of $M/N$. We change notation: we may replace $M$ by $M/N$, $u$ by its image in $M/N$, and $N$ by 0. Thus, we may assume that $u$ is in every nonzero submodule of $M$, and we want to show that $M$ has a unique associated prime. We also want to show that this prime is maximal. If $v \in M$ and $w \in M$ have distinct prime annihilators $P$ and $Q$, we have that $Rv \cong R/P$ and $Rw \cong R/Q$. Any nonzero element of $Rv \cap Rw$ has annihilator $P$ (thinking in $R/P$) and also has annihilator $Q$. It follows that $P = Q$ after all.

Thus, $\mathrm{Ass}\,(M)$ consists of a single prime ideal $P$. If $P$ is not maximal, we have an embedding $R/P \hookrightarrow M$. Then $u$ is in the image of $R/P$, and is in every nonzero ideal of $R/P$. If $R/P = D$ has dimension one or more, then it has a prime ideal $P'$ other than 0. Then $u$ must be in every power of $P'$, and so $u$ is in every power of the maximal ideal of the local ring $D_{P'}$, a contradiction. It follows that $\mathrm{Ass}\,(M)$ consists of a single maximal ideal $m$. This implies that $M$ has a finite filtration by copies of $R/m$, and is therefore killed by a power of $m$. Then $u$ must be in the socle $\mathrm{Ann}_M m$, which must be a one-dimensional vector space over $K = R/m$, or else it will have a subspace that does not contain $u$. $\square$

**Proposition (prime avoidance for cosets).** *Let $S$ be any commutative ring, $x \in S$, $I \subseteq S$ an ideal and $P_1, \ldots, P_k$ prime ideals of $S$. Suppose that the coset $x + I$ is contained in $\bigcup_{i=1}^k P_i$. Then there exists $j$ such that $Sx + I \subseteq P_j$.*

*Proof.* If $k = 1$ the result is clear. Choose $k \geq 2$ minimum giving a counterexample. Then no two $P_i$ are comparable, and $x + I$ is not contained in the union of any $k - 1$ of the $P_i$. Now $x = x + 0 \in x + I$, and so $x$ is in at least one of the $P_j$: say $x \in P_k$. If $I \subseteq P_k$, then $Sx + I \subseteq P_k$ and we are done. If not, choose $i_0 \in I - P_k$. We can also choose $i \in I$ such that $x + i \notin \bigcup_{j=1}^{k-1} P_i$. Choose $u_j \in P_j - P_k$ for $j < k$, and let $u$ be the product of the $u_j$. Then $ui_0 \in I - P_k$, but is in $P_j$ for $j < k$. It follows that $x + (i + ui_0) \in x + I$, but is not in any $P_j$, $1 \leq j \leq k$, a contradiction. $\square$

**Proposition.** *Let $R$ be a Noetherian ring and let $W$ be a multiplicative system. Then every element of $(W^{-1}R)^\circ$ has the form $c/w$ where $c \in R^\circ$ and $w \in W$.*

*Proof.* Suppose that $c/w \in (W^{-1}R)^\circ$ where $c \in R$ and $w \in W$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be the minimal primes of $R$ that do not meet $W$, so that the ideals $\mathfrak{p}_j W^{-1}R$ for $1 \leq j \leq k$ are *all* of the minimal primes of $W^{-1}R$. It follows that the image of $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$ is nilpotent in $W^{-1}R$, and so we can choose an integer $N > 0$ such that $I = (\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k)^N$ has image $0$ in $W^{-1}R$. If $c + I$ is contained in the union of the minimal primes of $R$, then by the coset form of prime avoidance above, it follows that $cR + I \subseteq \mathfrak{p}$ for some minimal prime $\mathfrak{p}$ of $R$. Since $I \subseteq \mathfrak{p}$, we have that $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k \subseteq \mathfrak{p}$, and it follows that $\mathfrak{p}_j = \mathfrak{p}$ for some $j$, where $1 \leq j \leq k$. But then $c \in \mathfrak{p}_j$, a contradiction, since $c/w$ and, hence, $c/1$, is not in any minimal prime of $R^\circ$. Thus, we can choose $g \in I$ such that $c + g$ is in $R^\circ$, and we have that $c/w = (c + g)/w$ since $g \in I$. $\square$

**Lemma.** *Let $R$ be a Noetherian ring of prime characteristic $p > 0$. Let $\mathfrak{A}$ be an ideal of $R$ primary to a maximal ideal $m$ of $R$. Then $\mathfrak{A}$ is tightly closed in $R$ if and only if $\mathfrak{A}R_m$ is tightly closed in $R_m$.*

*Proof.* Note that $R/\mathfrak{A}$ is already a local ring whose only maximal ideal is $m/\mathfrak{A}$. It follows that $(*)\quad R/\mathfrak{A} \cong (R/\mathfrak{A})_m = R_m/\mathfrak{A}R_m$. If $u \in R - \mathfrak{A}$ but $u \in \mathfrak{A}^*$, this is evidently preserved when we localize at $m$. Hence, if $\mathfrak{A}R_m$ is tightly closed in $R_m$, then $\mathfrak{A}$ is tightly closed in $R$. Now suppose $(\mathfrak{A}R_m)^*$ in $R_m$ contains an element not in $\mathfrak{A}R_m$. Without loss of generality, we may assume that this element has the form $f/1$ where $f \in R$. Suppose that $c_1 \in R_m^\circ$ has the property that $c_1 f^q \in \mathfrak{A}^{[q]}R_m = (\mathfrak{A}R_m)^{[q]}$ for all $q \gg 0$. By the preceding Proposition, $c_1$ has the form $c/w$ where $c \in R^\circ$ and $w \in R - m$. We may replace $c_1$ by $wc_1$, since $w$ is a unit, and therefore assume that $c_1 = c/1$ is the image of $c \in R^\circ$. Then $cf^q/1 \in \mathfrak{A}^{[q]}R_m$ for all $q \gg 0$. It follows from $(*)$ above that $cf^q \in \mathfrak{A}^{[q]}$ for all $q \gg 0$, and so $f \in \mathfrak{A}_R^*$, as required. $\square$

We have the following consequence:

**Theorem.** *Let $R$ be a Noetherian ring of prime characteristic $p > 0$. Then the following conditions are equivalent:*

(a) *$R$ is weakly F-regular.*

(b) *$R_m$ is weakly F-regular for every maximal ideal $m$ of $R$.*

(c) *Every ideal of $R$ primary to a maximal ideal of $R$ is tightly closed.*

*Proof.* It is clear that (a) $\Rightarrow$ (c). To see that (c) $\Rightarrow$ (a), assume (c) and suppose, to the contrary, that $u \in I^* - I$ in $R$. Let $\mathfrak{A}$ be maximal in $R$ with respect to the property of containing $I$ but not $u$. By the Lemma on p. 1, $R/\mathfrak{A}$ is killed by a power of a maximal ideal $m$, so that $\mathfrak{A}$ is $m$-primary. We still have $u \in \mathfrak{A}^* - \mathfrak{A}$, a contradiction. Then (b) holds if and only if all ideals primary to the maximal ideal of some $R_m$ are tightly closed, and the equivalence with (c) follows from the preceding Lemma. $\square$

We next make the following elementary observations about tight closure.

**Proposition.** *Let $R$ be a Noetherian ring of prime characteristic $p > 0$.*

(a) *The tight closure of $0$ in $R$ is the ideal $J$ of all nilpotent elements of $R$.*

(b) *For every ideal $I \subseteq R$, $I^* \subseteq \overline{I} \subseteq \mathrm{Rad}\,(I)$.*

(c) *Prime ideals, radical ideals, and integrally closed ideals are tightly closed in $R$.*

*Proof.* (a) If $cu^q = 0$ and $c$ is not in any minimal prime, then $u^q$ is in every minimal prime, and, hence, so is $u$. This shows that $0^* \subseteq J$. On the other hand, if $u$ is nilpotent, $u^{q_0} = 0$ for sufficiently large $q_0$, and then $1 \cdot u^q = 0$ for all $q \geq q_0$.

(b) Suppose $u \in I^*$. To show that $u \in \overline{I}$, it suffices to verify this modulo every minimal prime $P$ of $R$. When we pass to $R/P$, we still have that the image of $u$ is in the tight closure of $I(R/P)$. Hence, we may assume that $R$ is a domain. We then have $c \neq 0$ such that $cu^q \in I^{[q]} \subseteq I^q$ for all sufficiently large $q$, and, in particular, for infinitely many $q$. This is sufficient for $u \in \overline{I}$. If $u \in \overline{I}$, $u$ satisfies a monic polynomial

$$u^n + f_1 u^{n-1} + \cdots + f_n = 0$$

with $f_j \in I^J$ for $j \geq 1$. Thus, all terms but the first are in $I$, and so $u^n \in I$, which implies that $u \in \mathrm{Rad}\,(I)$.

(c) It is immediate from part (b) that integrally closed ideals are tightly closed in $R$, and that radical ideals are integrally closed. Of course, prime ideals are radical. $\square$

We next give a tight closure version of the Briançon-Skoda theorem. This result was proved by Briançon and Skoda [J. Briançon and H. Skoda, *Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de $C^n$*, C.R. Acad. Sci. Paris Sér. A **278** (1974) 949–951] for finitely generated $\mathbb{C}$-algebras and analytic regular local rings using a criterion of Skoda [H. Skoda, *Applications des techniques $L^2$ a la théorie des idéaux d'une algébre de fonctions holomorphes avec poids,* Ann. Scient. Ec. Norm. Sup. 4éme série, t. **5** (1972) 545–579] for when an analytic function is in an ideal in terms of the finiteness of a certain integral. Lipman and Teissier [J. Lipman and B. Teissier, *Pseudo-rational local rings and a theorem of Briançon-Skoda about integral closures of ideals,* Michigan Math. J. **28** (1981) 97–116] gave an algebraic proof for certain cases, and Lipman and

Sathaye [J. Lipman and A. Sathaye, *Jacobian ideals and a theorem of Briançon-Skoda*, Michigan Math. J. **28** (1981) 199–222] proved the result in general for regular rings. A detailed treatment of the Lipman-Sathaye argument is given in the Lecture Notes from Math 711, Fall 2006: see particularly the Lectures of September 25, 27, and 29, as well as the Lectures of October 2, 4, 6, 9, 11, and 13.

Tight closure gives an unbelievably simple proof of the theorem that is more general than these results in the equicharacteristic case, but the Lipman-Sathaye argument is the only one that is valid in mixed characteristic. Notice that in the tight closure version of the Theorem just below, the first statement is valid for *any* Noetherian ring of prime characteristic $p > 0$.

**Theorem (Briançon-Skoda).** *Let $R$ be a Noetherian ring of prime characteristic $p > 0$. Let $I$ be an ideal of $R$ that is generated by $n$ elements. Then $\overline{I^n} \subseteq I^*$. Hence, if $R$ is regular (or weakly F-regular) then $\overline{I^n} \subseteq I$.*

*Proof.* We may work modulo each minimal prime in turn, and so assume that $R$ is a domain. If $u \in \overline{I^n}$ there exists $c \neq 0$ such that for all $k \gg 0$, $cu^k \in (I^n)^k = I^{nk}$. In particular, this is true when $k = q = p^e$. The ideal $I^{nq} = (f_1, \, \ldots \, , f_n)^{nq}$ is generated by the monomials $f_1^{a_1} \cdots f_n^{a_n}$ of degree $nq$ in the $f_j$. But when $a_1 + \cdots + a_q = nq$, at least one of the $a_i$ is $\geq q$: if all are $\leq q-1$, their sum is $\leq n(q-1) < nq$. Thus, $I^{nq} \subseteq I^{[q]}$, and we have that $cu^q \in I^{[q]}$ for all $q \gg 0$. This shows that $u \in I^*$. The final statement holds because all ideals of a regular ring are tightly closed, $\square$

The Briançon-Skoda Theorem is often stated in a stronger but more technical form. The hypothesis is the same: $I$ is an ideal generated by $n$ elements. The conclusion is that $\overline{I^{n+m-1}} \subseteq (I^m)^*$ for all integers $m \geq 1$. The version we stated first is the case where $m = 1$. The argument for the strengthened version is very similar, but slightly more technical. Again, we may assume that $R$ is a domain and that $cu^q \subseteq (I^{n+m-1})^q$ for all $q \gg 0$. Consider a monomial $f_1^{a_1} \cdots f_n^{a_n}$ where the sum of the $a_i$ is $(n+m-1)q$. We can write each $a_i = b_i q + r_i$, where $0 \leq r_i \leq q-1$. It will suffice to show that the sum of the $b_i$ is at least $m$, for then the monomial is in $(I^m)^{[q]}$, and we have that $u \in (I^m)^*$. But if the sum of the $b_i$ is at most $m-1$, then the sum of the $a_i$ is bounded by $(m-1)q + n(q-1) = (n+m-1)q - n < (n+m-1)q$, a contradiction. $\square$

The equal characteristic 0 form of the Theorem can be deduced from the characteristic $p$ form by standard methods of reduction to characteristic $p$.

The basic tight closure form of the Briançon-Skoda theorem is of interest even in the case where $n = 1$, which has the following consequence.

**Proposition.** *Let $R$ be a Noetherian ring of prime characteristic $p > 0$. The tight closure of the principal ideal $I = fR$ is the same as its integral closure.*

*Proof.* By the Briançon-Skoda theorem when $n = 1$, we have that $\overline{I} \subseteq I^*$, while the other inclusion always holds. $\square$

We next observe:

**Theorem.** *Let $R$ be a Noetherian ring of prime characteristic $p > 0$. If the ideal $(0)$ and the principal ideals generated by nonzerodivisors are tightly closed, then $R$ is normal. Thus, if every principal ideal of $R$ is tightly closed, then $R$ is normal. Consequently, weakly F-regular rings are normal.*

*Proof.* The hypothesis that $(0)$ is tightly closed is equivalent to the assumption that $R$ is reduced. Henceforth, we assume that $R$ is reduced.

If $R$ is a product $S \times T$ then the hypothesis on $R$ holds in both factors. E.g., if $s$ is a nonzerodivisor in $s$, then $(s,\,1)$ is a nonzerodivisor in $T$: it generates the ideal $sS \times T$, and its tight closure in $S \times T$ is $(sS)^*_S \times T$. But this is the same as $sS \times T$ if and only if $sS$ is tightly closed in $S$.

Therefore, we may assume that $R$ is not a product, i.e., that $\mathrm{Spec}\,(R)$ is connected. We first want to show that $R$ is a domain in this case. If not, there are minimal primes $P_1,\,\ldots\,,P_n$, $n \geq 2$, and we can choose an element $u_i$ in $P_i - \bigcup_{j \neq i} P_j$ for every $i$. Let $u = u_1$, which is in $P_1$ and no other minimal prime, and $v = u_2 \cdots u_n$, which is in $P_2 \cap \cdots \cap P_n$ and not in $P_1$. Then $uv$ is in every minimal prime, and so is 0, while $f = u + v$ is a not in any minimal prime, and so is not a zerodivisor. We claim that $u \in (fR)^*$. It suffices to check this modulo every $P_i$. But mod $P_1$, $u \equiv 0 = 0 \cdot f$, and mod $P_j$ for $j > 1$, $u \equiv f = 1 \cdot f$. Since $(fR)^* = fR$, we can write $u = e(u + v)$ for some element $e \in R$. This means that $(1 - e)u = ev$. Mod $P_1$, $u \equiv 0$ while $v \not\equiv 0$, and so $e \equiv 0$ mod $P_1$. Mod $P_j$ for $j > 1$, $u \not\equiv 0$ while $v \equiv 0$, and so $e \equiv 1$ mod $P_j$. It follows that $e^2 - e$ is in every minimal prime, and so is 0. Since whether its value mod $P_i$ is 0 or 1 depends on $i$, $e$ is a non-trivial idempotent in $R$, a contradiction.

Thus, we may assume that $R$ is a domain. Now suppose that $f,\, g \in R$ with $g \neq 0$ and that $f/g$ is integral over $R$. Then we have an equation of integral dependence

$$(f/g)^s + r_1(f/g)^{s-1} + \cdots + r_j(f/g)^{s-j} + \cdots + r_s = 0$$

with the $r_j \in R$. Multiplying by $g^s$ we obtain

$$f^s + (r_1 g)f^{s-1} + \cdots (r_j g^j)f^{s-j} + \cdots + r_s g^s = 0,$$

which shows that $f$ is in the integral closure of $gR$. Thus, $f \in (gR)^*$, and this is $gR$ by hypothesis. Consequently, $f = gr$ with $r \in R$, which shows that $f/g = r \in R$, as required. $\square$

We next want to discuss the use of tight closure to prove Theorems about the behavior of symbolic powers in regular rings of prime characteristic $p > 0$. The characteristic $p$

results imply corresponding results in equal characteristic 0. The following result was first proved in equal characteristic 0 by Ein, Lazarsfeld, and Smith [L. Ein, R. Lazarsfeld, and K. E. Smith, *Uniform bounds and symbolic powers on smooth varieties*, Inventiones Math. **144** (2001) 241–252], using the theory of multiplier ideals. The proof we give here may be found in [M. Hochster and C. Huneke, *Comparison of symbolic and ordinary powers of ideals*, Inventiones Math. **147** (2002) 349–369].

**Theorem.** *Let $P$ be a prime ideal of height $h$ in a regular ring $R$ of prime characteristic $p > 0$. Then for every integer $n \geq 1$, $P^{(hn)} \subseteq P^n$.*

There are sharper results if one places additional hypotheses on $R/P$. An extreme example is to assume that $R/P$ is regular so that, locally, $P$ is generated by a regular sequence. In this case, the symbolic and ordinary powers of $P$ are equal. Doubtless the best results of this sort remain to be discovered. It is not known whether the conclusion of the Theorem above holds in regular rings of mixed characteristic. The version stated above remains true with the hypotheses weakened in various ways. There are further comments about what can be proved in the sequel: see the last paragraph on p. 7. We have attempted to give a result that is of substantial interest but that has relatively few technicalities in its proof. The methods used here also yield the result that, without any regularity hypothesis on $R$, if $R/P$ has finite projective dimension over $R$ then

$$P^{(hn)} \subseteq (P^n)^*.$$

Of course, if $R$ is regular the hypothesis of finite projective dimension is automatic, while one does not need to take the tight closure on the right because, in a regular ring, every ideal is tightly closed.

We postpone the proof of the Theorem to give a preliminary result that we will need.

**Lemma.** *Let $P$ be a prime ideal of height $h$ in a regular ring $R$ of prime characteristic $p > 0$.*

(a) $P^{[q]}$ *is primary to $P$.*

(b) $P^{(qh)} \subseteq P^{[q]}$.

*Proof.* For part (a), we have that $\mathrm{Rad}\,(P^{[q]}) = P$, clearly. Let $f \in R - P$. It suffices to show that $f$ is not a zerodivisor on $R/P^{[q]}$. Since

$$0 \to R/P \xrightarrow{f\cdot} R/P$$

is exact, it remains exact when we tensor with $R$ viewed as an $R$-algebra via $F^e$, since this is a flat base change. Thus,

$$0 \to \mathcal{F}^e(R/P) \xrightarrow{f^q\cdot} \mathcal{F}^e(R/P)$$

is exact, and this is

$$0 \to R/P^{[q]} \xrightarrow{f^q \cdot} R/P^{[q]}.$$

Since $f^q$ is not a zerodivisor on $R/P^{[q]}$, neither is $f$.

Suppose $u \in P^{(qh)} - P^{[q]}$. Make a base change to $R_P$. Then the image of $u$ is in $P^{qh}R_P$, but not in $P^{[q]}R_P = (PR_P)^{[q]}$: if $u$ were in the expansion of $P^{[q]}R_P$, it would be multiplied into $P^{[q]}$ by some element of $R - P$. Since such an element is not in $P^{[q]}$ by part (a), we have $u \notin (PR_P)^{[q]}$. But $PR_P$ is generated by $h$ elements, and so

$$(PR_P)^{qh} \subseteq (PR_P)^{[q]}$$

exactly as in the proof of the Briançon-Skoda Theorem: if a monomial in $h$ elements has degree $qh$, at least one of the exponents occurring on one of the elements must be at least $q$. $\square$

*Proof of the symbolic power theorem.* If $u \in P^{(hn)} - P^n$, then this continues to be the case after localizing at a maximal ideal in the support of $(P^n + Ru)/P^n$. Hence, we may assume that $R$ is regular local. We may also assume that $P \neq 0$. Given $q = p^e$ we can write $q = an + r$ where $a \geq 0$ and $0 \leq r \leq n - 1$ are integers. Then $u^a \in P^{(han)}$ and

$$P^{hn}u^a \subseteq P^{hr}u^a \subseteq P^{(han+hr)} = P^{(hq)} \subseteq P^{[q]}.$$

Taking $n$ th powers gives that

$$P^{hn^2}u^{an} \subseteq (P^{[q]})^n = (P^n)^{[q]},$$

and since $q \geq an$, we have that

$$P^{hn^2}u^q \subseteq (P^n)^{[q]}$$

for fixed $h$ and $n$ and for all $q$. Let $d$ be any nonzero element of $P^{hn^2}$. The condition that $du^q \in (P^n)^{[q]}$ for all $q$ says precisely that $u$ is in the tight closure of $P^n$ in $R$. But in a regular ring, every ideal is tightly closed, and so $u \in P^n$, as required. $\square$

One can prove a similar result for ideals $I$ without assuming that $I$ is prime and without assuming that the ring is regular. We can define symbolic powers of ideals that are not necessarily prime as follows. If $W$ is the multiplicative system of nonzerodivisors on $I$, define $I^{(t)}$ as the contraction of $I^tW^{-1}R$ to $R$. Suppose that $R/I$ has finite projective dimension over $R$ and that the localization of $I$ at any associated prime of $I$ can be generated by at most $h$ elements (or even that its analytic spread is at most $h$). Then one can show $I^{(nh)} \subseteq (I^n)^*$ for all $n \geq 1$. See Theorem (1.1) of [M. Hochster and C. Huneke, *Comparison of symbolic and ordinary powers of ideals*, Inventiones Math. **147** (2002) 349–369].

**Test elements**

The definition of tight closure allows the element $c \in R^\circ$ to vary with $N$, $M$, and the element $u \in M$ being "tested" for membership in $N^*_M$. But under mild conditions on a reduced ring $R$, there exist elements, called *test elements*, that can be used in every tight closure test. It is somewhat difficult to prove their existence, but they play a very important role in the theory of tight closure.

**Definition.** Let $R$ be a Noetherian ring of prime characteristic $p > 0$. An element $c \in R^\circ$ is called a *test element* (respectively, *big test element*) for $R$ if for every inclusion of finitely generated modules $N \subseteq M$ (respectively, arbitrary modules $N \subseteq M$) and every $u \in M$, $u \in N^*_M$ if and only if $cu^q \in N^{[q]}_M$ for every $q = p^e \geq 1$. A (big) test element is called *locally stable* if it is a (big) test element in every localization of $R$. A (big) test element is called *completely stable* if it is a (big) test element in the completion of every local ring of $R$.

It will be a while before we can prove that test elements exist. But we shall eventually prove the following:

**Theorem.** *Let $R$ be a Noetherian ring of prime characteristic $p > 0$ that is reduced and essentially of finite type over an excellent semilocal ring $R$. Let $c \in R^\circ$ be such that $R_c$ is regular (such elements always exist). Then $c$ has a power that is a completely stable big test element for $R$.*

We want to record some easy facts related to test elements. We first note:

**Lemma.** *If $N \subseteq M$ are $R$-modules, $S$ is faithfully flat over $R$, and $v \in M - N$, then $1 \otimes v$ is not in $\langle S \otimes_R N \rangle$ in $S \otimes_R M$.*

*Proof.* We may replace $M$ by $M/N$, $N$ by $0$, and $v$ by its image in $M/N$. The result then asserts that the map $M \to S \otimes_R M$ is injective. Let $v \in M$ be in the kernel. Then $S \otimes_R Rv \hookrightarrow S \otimes_R M$, and it suffices to see that $(*)\quad Rv \to S \otimes_R Rv$ is injective. Let $I = \mathrm{Ann}_R v$. Then $(*)$ is equivalent to the assertion that $R/I \to S/IS$ is injective. Since $S/IS$ is faithfully flat over $R/I$, we need only show that if $R \to S$ is faithfully flat, it is injective. Let $J \subseteq R$ be the kernel. Then $J \otimes S \cong JS = 0$, which implies that $J = 0$. $\square$

**Proposition.** *Let $R$ be a Noetherian ring of prime characteristic $p > 0$, and let $c \in R$.*

(a) *If for every pair of modules (respectively, finitely generated modules) $N \subseteq M$ one has $cN^*_M \subseteq N$, then one also has that whenever $u \in N^*_M$, then $cu^q \in N^{[q]}_M$ for all $q$. Thus, $c$ is a big test element (respectively, test element) for $R$ if and only if $c \in R^\circ$ and $cN^*_M \subseteq N$ for all inclusions of modules (respectively, finitely generated modules) $N \subseteq M$.*

(b) *If $c \in R^0$, $S$ is faithfully flat over $R$, and $c$ is (big) test element for $S$, then it is a (big) test element for $R$. If $c$ is a completely stable (big) test element for $S$, then $c$ is a completely stable (big) test element for $S$.*

(c) *If the image of* $c \in R^\circ$ *is a (big) test element in* $R_m$ *for every maximal ideal* $m$ *of* $R$*, then* $c$ *is a test element for* $R$*.*

(d) *If* $c \in R^\circ$ *and* $c$ *is a (big) test element for* $R_P$ *for every prime ideal* $P$ *of* $R$*, then* $c$ *is a (big) test element for* $W^{-1}R$ *for every multiplicative system* $W$ *of* $R$*, i.e.,* $c$ *is a locally stable (big) test element for* $R$*.*

(e) *If* $c$ *is a completely stable (big) test element for* $R$ *then it is a locally stable (big) test element for* $R$*.*

*Proof.* In each part, if we are proving a statement about test elements we assume that $N \subseteq M$ are finitely generated, while if we are proving a statement about big test elements, we allow them to be arbitrary.

(a) If $u \in N^*_M$ we also have that $u^q \in (N^{[q]})^*_{\mathcal{F}^e(M)}$ for all $q$, and hence that $cu^q \in N^{[q]}$, as required.

(b) Suppose that $u \in N^*_M$. Then $1 \otimes u$ is in $\langle S \otimes_R N \rangle^*$ in $S \otimes_R M$, and it follows that $c(1 \otimes u) = 1 \otimes cu$ is in $\langle S \otimes_R N \rangle$ in $S \otimes_R M$. Because $S$ is faithfully flat over $R$, it follows from the preceding Lemma that $cu \in N$. The second statement follows from the first, because of $P$ is prime in $R$ and $Q$ is a minimal prime of $PS$, then $R_P \to S_Q$ is faithfully flat, and hence so is the induced map of completions $\widehat{R_P} \to \widehat{S_Q}$. Since $c$ is a (big) test element for $\widehat{S_Q}$, it is a (big) test element for $\widehat{R_P}$.

(c) Suppose that $u \in N^*_M$ in $R$. If $cu \notin N$, then there exists a maximal ideal $m$ in the support of $(N + Rcu)/N$. When we pass to $R_m$, $N_m \subseteq M_m$, and $u/1$, the image of $u$ in $M$, we still have that $u/1$ is in $(N^m)^*_{M_m}$ working over $R_m$. If follows that $cu/1 \in N_m$, a contradiction.

(d) follows from (c), because every localization of $W^{-1}R$ at a maximal ideal is a localization of $R$ at some prime ideal $P$.

(e) follows from (d) and (b), because for every prime ideal $P$ of $R$, the completion of $R_P$ is faithfully flat over $R_P$. $\square$

**Definition: test ideals.** Let $R$ be a Noetherian ring of prime characteristic $p > 0$, and assume that $R$ is reduced. We define $\tau(R)$ to be the set of elements $c \in R$ such that $cN^*_M \subseteq N$ for all inclusion maps $N \subseteq M$ of finitely generated $R$-modules. Alternatively, we may write:

$$\tau(R) = \bigcap_{N \subseteq M \text{ finitely generated}} N :_R N^*_M,$$

and we also have that

$$N :_R N^*_M = \mathrm{Ann}_R(N^*_M/N).$$

We refer $\tau(R)$ as the *test ideal* of $R$.

We define $\tau_{\mathrm{b}}(R)$ to be the set of elements $c \in R$ such that $cN^*_M \subseteq N$ for all inclusion maps $N \subseteq M$ of arbitrary $R$-modules. Alternatively, we may write:

$$\tau_{\mathrm{b}}(R) = \bigcap_{N \subseteq M} N :_R N^*_M,$$

and refer to $\tau_{\mathrm{b}}(R)$ as the *big test ideal* of $R$, although it is obviously contained in $\tau(R)$. We shall see below that if $R$ has a (big) test element, then $\tau(R)$ (respectively, $\tau_{\mathrm{b}}(R)$) is generated by all the (big) test elements of $R$. We first note:

**Lemma.** *Let $R$ be any ring and $P_1, \ldots, P_k$ any finite set of primes of $R$. Let*

$$W = R - \bigcup_{i=1}^{k} P_i.$$

*If an ideal $I$ of $R$ is not contained in any of the $P_j$, then $I$ is generated by its intersection with $W$. In particular, if $R$ is Noetherian and $I$ is not contained in any minimal prime of $R$, then $I$ is generated by its intersection with $R^\circ$.*

*Proof.* Let $J$ be the ideal generated by all elements of $I \cap W$. Then

$$I \subseteq J \cup P_1 \cup \cdots \cup P_k,$$

since every element of $I$ not in any of the $P_i$ is in $J$. Since all but one of the ideals on the right is prime, we have that $I \subseteq J$ or $I \subseteq P_i$ for some $i$. Since $I$ contains at least one element of $W$, it is not contained in any of the $P_i$. Thus, $J \subseteq I \subseteq J$, and so $J = I$, as required. The final statement now follows because a Noetherian ring has only finitely many minimal primes. $\square$

**Proposition.** *Let $R$ be a Noetherian ring of prime characteristic $p > 0$, and assume that $R$ is reduced.*

(a) $\tau_{\mathrm{b}}(R) \subseteq \tau(R)$.

(b) *$\tau(R) \cap R^\circ$ (respectively, $\tau_{\mathrm{b}}(R) \cap R^\circ$) is the set of test elements (respectively, big test elements) of $R$.*

(c) *If $R$ has at least one test element (respectively, one big test element), then $\tau(R)$ (respectively, $\tau_{\mathrm{b}}(R)$) is the ideal of $R$ generated by all test elements (respectively, all big test elements) of $R$.*

*Proof.* (a) is clear from the definition, and so is (b). Part (c) then follows from the preceding Lemma. $\square$