

Math 711: Lecture of September 19, 2007

Earlier (see the Lecture of September 7, p. 7) we discussed very briefly the class of excellent Noetherian rings. The condition that a ring be excellent or, at least, locally excellent, is the right hypothesis for many theorems on tight closure. The theory of excellent rings is substantial enough to occupy an entire course, and we do not want to spend an inordinate amount of time on it here. We shall summarize what we need to know about excellent rings in this lecture. In the sequel, the reader who prefers may restrict attention to rings essentially of finite type over a field or over a complete local ring, which is the most important family of rings for applications. The definition of an excellent Noetherian ring was given by Grothendieck. A readable treatment of the subject, which is a reference for all of the facts about excellent rings stated without proof in this lecture, is [H. Matsumura, *Commutative Algebra*, W.A. Benjamin, New York, 1970], Chapter 13.

Before discussing excellence, we want to review the notion of fibers of ring homomorphisms.

Fibers

Let $f : R \rightarrow S$ be a ring homomorphism and let P be a prime ideal of R . We write κ_P for the canonically isomorphic R -algebras

$$\text{frac}(R/P) \cong R_P/PR_P.$$

By the *fiber* of f over P we mean the κ_P -algebra

$$\kappa_P \otimes_R S \cong (R - P)^{-1}S/PS$$

which is also an R -algebra (since we have $R \rightarrow \kappa_P$) and an S -algebra. One of the key points about this terminology is that the map

$$\text{Spec}(\kappa_P \otimes_R S) \rightarrow \text{Spec}(S)$$

gives a bijection between the prime ideals of $\kappa_P \otimes_R S$ and the prime ideals of S that lie over $P \subseteq R$. In fact, it is straightforward to check that $\text{Spec}(\kappa_P \otimes_R S)$ is homeomorphic with its image in $\text{Spec}(S)$.

It is also said that $\text{Spec}(\kappa_P \otimes_R S)$ is the *scheme-theoretic* fiber of the map

$$\text{Spec}(S) \rightarrow \text{Spec}(R).$$

This is entirely consistent with thinking of the fiber of a map of sets $g : Y \rightarrow X$ over a point $P \in X$ as

$$g^{-1}(P) = \{Q \in Y : g(Q) = P\}.$$

In our case, we may take $g = \text{Spec}(f)$, $Y = \text{Spec}(S)$, and $X = \text{Spec}(R)$, and then $\text{Spec}(\kappa_P \otimes_R S)$ may be naturally identified with the set-theoretic fiber of

$$\text{Spec}(S) \rightarrow \text{Spec}(R).$$

If R is a domain, the fiber over the prime ideal (0) of R , namely $\text{frac}(R) \otimes_R S$, is called the *generic fiber* of $R \rightarrow S$.

If (R, m, K) is quasilocal, the fiber $K \otimes_R S = S/mS$ over the unique closed point m of $\text{Spec}(R)$ is called the *closed fiber* of $R \rightarrow S$.

Geometric regularity

Let κ be a field. A Noetherian κ -algebra R , is called *geometrically regular* over κ if the following two equivalent conditions hold:

- (1) For every finite algebraic field extension κ' of κ , $\kappa' \otimes_\kappa R$ is regular.
- (2) For every finite purely inseparable field extension κ' of κ , $\kappa' \otimes_\kappa R$ is regular.

Of course, since we may take $\kappa' = \kappa$, if R is geometrically regular over κ then it is regular. In equal characteristic 0, geometric regularity is equivalent to regularity, using characterization (2).

When R is essentially of finite type over κ , these conditions are also equivalent to

- (3) $K \otimes_\kappa R$ is regular for every field K
- (4) $K \otimes_\kappa R$ is regular for one perfect field extension K of κ .
- (5) $K \otimes_\kappa R$ is regular when $K = \bar{\kappa}$ is the algebraic closure of κ .

These conditions are not equivalent to (1) and (2) in general, because $K \otimes_\kappa R$ need not be Noetherian.

We indicate how the equivalences are proved. This will require a very considerable effort.

Theorem. *Let $R \rightarrow S$ be a faithfully flat homomorphism of Noetherian rings. If S is regular, then R is regular.*

Proof. We use the fact that a local ring A is regular if and only its residue class field has finite projective dimension over A , in which case every finitely generated module has finite projective dimension over A . Given a prime P of R , there is a prime Q of S lying over it. It suffices to show that R_P is regular, and we have a faithfully flat map $R_P \rightarrow S_Q$. Therefore we may assume that $(R, P, K) \rightarrow (S, Q, L)$ is a flat, local homomorphism and

that S is regular. Consider a minimal free resolution of R/P over R , which, *a priori*, may be infinite:

$$\dots \rightarrow R^{b_n} \xrightarrow{\alpha_n} R^{b_{n-1}} \rightarrow \dots \xrightarrow{\alpha_1} R^{b_0} \rightarrow R/P \rightarrow 0.$$

By the minimality of the resolution, the matrices α_j all have entries in P . Now apply $S \otimes_R _$. We obtain a free resolution

$$\dots \rightarrow S^{b_n} \xrightarrow{\alpha_n} S^{b_{n-1}} \rightarrow \dots \xrightarrow{\alpha_1} S^{b_0} \rightarrow S \otimes_R S/PS \rightarrow 0,$$

where we have identified R with its image in S under the injection $R \hookrightarrow S$. This resolution of S/PS is minimal: the matrices have entries in Q because $R \hookrightarrow S$ is local. Since S is regular, S/PS has finite projective dimension over S , and so the matrices α_j must be 0 for all $j \gg 0$. But this implies that the projective dimension of R/P over R is finite. \square

Corollary. *If R is a Noetherian K -algebra and L is an extension field of K such that $L \otimes_K R$ is regular (in general, this ring may not be Noetherian, although it is if R is essentially of finite type over K , because in that case $L \otimes_K R$ is essentially of finite type over L , and therefore Noetherian), then R is regular.*

Proof. Since L is free over K , it is faithfully flat over K , and so $L \otimes_K R$ is faithfully flat over R and we may apply the preceding result. \square

Proposition. *Let $(R, m, K) \rightarrow (S, Q, L)$ be a flat local homomorphism of local rings. Then*

- (a) $\dim(S) = \dim(R) + \dim(S/mS)$, the sum of the dimensions of the base and of the closed fiber.
- (b) If R is regular and S/mS is regular, then S is regular.

Proof. (a) We use induction on $\dim(R)$. If $\dim(R) = 0$, m and mS are nilpotent. Then $\dim(S) = \dim(S/mS) = \dim(R) + \dim(S/mS)$, as required. If $\dim(R) > 0$, let J be the ideal of nilpotent elements in R . Then $\dim(R/J) = \dim(R)$, $\dim(S/JS) = \dim(S)$, and the closed fiber of $R/J \rightarrow S/JS$, which is still a flat and local homomorphism, is S/mS . Therefore, we may consider the map $R/J \rightarrow S/JS$ instead, and so we may assume that R is reduced. Since $\dim(R) > 0$, there is an element $f \in m$ not in any minimal prime of R , and, since R is reduced, f is not in any associated prime of R , i.e., f is a nonzerodivisor in R . Then the fact that S is flat over R implies that f is not a zerodivisor in S . We may apply the induction hypothesis to $R/fR \rightarrow S/fS$, and so

$$\dim(S) - 1 = \dim(S/fS) = \dim(R/f) + \dim(S/mS) = \dim(R) - 1 + \dim(S/mS),$$

and the result follows.

(b) The least number of generators of Q is at most the sum of the number of generators of m and the number of generators of Q/mS , i.e., it is bounded by $\dim(R) + \dim(S/mS) = \dim(S)$ by part (a). The other inequality always holds, and so S is regular. \square

Corollary. *Let $R \rightarrow S$ be a flat homomorphism of Noetherian rings. If R is regular and the fibers of $R \rightarrow S$ are regular, then S is regular.*

Proof. If Q is any prime of S we may apply part (b) of the preceding Theorem, since S_Q/PS_Q is a localization of the fiber $\kappa_P \otimes_R S$, and therefore regular. \square

Corollary. *Let R be a regular Noetherian K -algebra, where K is a field, and let L be a separable extension field of K such that $L \otimes_K R$ is Noetherian. Then $L \otimes_K R$ is regular.*

Proof. The extension is flat, and so it suffices to show that every $\kappa_P \otimes_R (L \otimes_K R) \cong \kappa_P \otimes_K L$ is regular. Since L is algebraic over K , this ring is integral over κ_P and so zero-dimensional. Since $L \otimes_K R$ is Noetherian by hypothesis, $\kappa_P \otimes_K L$ is Noetherian, and so has finitely many minimal primes. Hence, it is Artinian, and if it is reduced, it is a product of fields and, therefore, regular as required. Thus, it suffices to show that $\kappa_P \otimes_K L$ is reduced. Since L is a direct limit of finite separable algebraic extension, it suffices to prove the result when L is a finite separable extension of K . In this case, L has a primitive element θ , and $L \cong K[x]/g$ where $g \in K[x]$ is a monic irreducible separable polynomial over $K \subseteq \kappa_P$. Let Ω denote the algebraic closure of κ_P . Then $\kappa_P \otimes_K L \subseteq \Omega \otimes_K L$, and so it suffices to show that

$$\Omega \otimes_K L \cong \Omega \otimes_K (K[x]/gK[x]) \cong \Omega[x]/g\Omega[x]$$

is reduced. This follows because g is separable, and so has distinct roots in Ω . \square

Theorem. *Let K be an algebraically closed field and let L be any finitely generated field extension of K . Then L has a separating transcendence basis \mathcal{B} , i.e., a transcendence basis \mathcal{B} such that L is separable over the pure transcendental extension $K(\mathcal{B})$.*

Proof. If F is a subfield of L , let F^{sep} denote the separable closure of F in L . Choose a transcendence basis x_1, \dots, x_n so as to minimize $[L : L']$ where $L' = K(x_1, \dots, x_n)^{\text{sep}}$. Suppose that $y \in L$ is not separable over $K(x_1, \dots, x_n)$. Choose a minimal polynomial $F(z)$ for y over $K(x_1, \dots, x_n)$. Then every exponent on z is divisible by p . Put each coefficient in lowest terms, and multiply $F(z)$ by a least common multiple of the denominators of the coefficients. This yields a polynomial $H(x_1, \dots, x_n, z) \in K[x_1, \dots, x_n][z]$ such that the coefficients in $K[x_1, \dots, x_n]$ are relatively prime, and such that the polynomial is irreducible over $K(x_1, \dots, x_n)[z]$. By Gauss's Lemma, this polynomial is irreducible in $K[x_1, \dots, x_n, z]$. It cannot be the case that every exponent on every x_j is divisible by p , for if that were true, since the field is perfect, H would be a p th power, and not irreducible. By renumbering the x_i we may assume that x_n occurs with an exponent not divisible by p . Then the element x_n is separable algebraic over the field $K(x_1, \dots, x_{n-1}, y)$, and we may use the transcendence basis x_1, \dots, x_{n-1}, y for L . Note that $x_n, y \in K(x_1, \dots, x_{n-1}, y)^{\text{sep}} = L''$, which is therefore strictly larger than $L' = K(x_1, \dots, x_n)^{\text{sep}}$. Hence, $[L : L''] < [L : L']$, a contradiction. \square

We can now prove:

Theorem. *Let R be a Noetherian κ -algebra, where κ is a field. Then the following two conditions are equivalent:*

- (1) *For every finite algebraic field extension κ' of κ , $\kappa' \otimes_{\kappa} R$ is regular.*
- (2) *For every finite purely inseparable field extension κ' of κ , $\kappa' \otimes_{\kappa} R$ is regular.*

Moreover, if R is essentially of finite type over κ then the following three conditions are equivalent to (1) and (2) as well:

- (3) *$K \otimes_{\kappa} R$ is regular for every field K*
- (4) *$K \otimes_{\kappa} R$ is regular for one perfect field extension K of κ .*
- (5) *$K \otimes_{\kappa} R$ is regular when $K = \bar{\kappa}$ is the algebraic closure of κ .*

Proof. We shall repeatedly use that if we have regularity for a larger field extension, then we also have it for a smaller one: this follows from the Corollary on p. 3.

Evidently, (1) \Rightarrow (2). But (2) \Rightarrow (1) as well, because given any finite algebraic extension κ' of κ , there is a larger finite field extension obtained by first making a finite purely inseparable extension and then a finite separable extension. The purely inseparable extension yields a regular ring by hypothesis, and the separable field extension yields a regular ring by the second Corollary on p. 4.

Now consider the case where R is essentially of finite type over κ . Evidently, (3) \Rightarrow (5) \Rightarrow (4) \Rightarrow (2) (the last holds because any perfect field extension contains the perfect closure, and this contains every finite purely inseparable algebraic extension), and it will suffice to prove that (2) \Rightarrow (3).

Let κ^{∞} denote the perfect closure $\bigcup_q \kappa^{1/q}$ of κ . We first show that $\kappa^{\infty} \otimes_{\kappa} R$ is regular.

Replace R by R_m . Then $B = \kappa^{\infty} \otimes_R R_m$ is purely inseparable over R_m : consequently, it is a local ring of the same dimension as R_m , and it is the directed union of the local rings $\kappa' \otimes_{\kappa} R_m$ as κ' runs through finite purely inseparable extensions of κ contained in κ^{∞} . All of these local rings have the same dimension: call it d . Let u_1, \dots, u_n be a minimal set of generators of the maximal ideal of $B = \kappa^{\infty} \otimes_{\kappa} R_m$, and choose κ' sufficiently large that u_1, \dots, u_n are elements of $A = \kappa' \otimes_R R_m$. Let $J = (u_1, \dots, u_n)A$. Since B is faithfully flat over A , we have that $JB \cap A = J$. But JB is the maximal ideal of B , which lies over the maximal ideal of A , and so J generates the maximal ideal of A . None of the generators is an A -linear combination of the others, or else this would also be true in B . Hence, u_1, \dots, u_n is a minimal set of generators of the maximal ideal of A . Since A is regular, $n = d$, and so B is regular.

Since the algebraic closure of κ is separable over κ^{∞} , it follows from the second Corollary on p. 4 that (2) \Rightarrow (5). To complete the proof, it suffices to show that if κ is algebraically closed, R is regular, and L is any field extension of κ , then $L \otimes_{\kappa} R$ is regular. Since $R \rightarrow L \otimes_{\kappa} R$ is flat, it suffices to show the fibers $L \otimes_{\kappa} \kappa_P$ are regular, and κ_P is finitely generated as a field over κ . Hence, κ_P has a separating transcendence basis x_1, \dots, x_n

over κ . Let $K = \kappa(x_1, \dots, x_n)$. Then

$$L \otimes_{\kappa} \kappa_P = (L \otimes_{\kappa} \kappa(x_1, \dots, x_n)) \otimes_K \kappa_P.$$

Since κ_P is a finite separable algebraic extension of K , it suffices prove that $L \otimes_{\kappa} K$ is regular. But this ring is a localization of $L[x_1, \dots, x_n]$, and so the proof is complete. \square

We say that a homomorphism $R \rightarrow S$ of Noetherian rings is *geometrically regular* if it is flat and all the fibers $\kappa_P \rightarrow \kappa_P \otimes_R S$ are geometrically regular. (Some authors use the term “regular” for this property.)

For those readers familiar with *smooth homomorphisms*, we mention that if S is essentially of finite type over R , then S is geometrically regular if and only if it is smooth.

By a very deep result of Popescu (cf. [D. Popescu, *General Néron desingularization*, Nagoya Math. J. **100** (1985) 97–126], every geometrically regular map is a direct limit of smooth maps. Whether Popescu’s argument was correct was controversial for a while. Richard Swan showed that Popescu’s argument was essentially correct in [R. G Swan, *Néron-Popescu desingularization*, Algebra and geometry (Taipei, 1995), 135–192, Lect. Algebra Geom. **2** Int. Press, Cambridge, MA, 1998].

Catenary and universally catenary rings

A Noetherian ring is called *catenary* if for any two prime ideals $P \subseteq Q$, any two saturated chains of primes joining P to Q have the same length. In this case, the common length will be the same as the dimension of the local domain R_Q/PR_Q .

Nagata was the first to give examples of Noetherian rings that are not catenary. E.g., in [M. Nagata, *Local Rings*, Interscience, New York, 1962] Appendix, pp. 204–5, Nagata gives an example of a local domain (D, m) of dimension 3 containing a height one prime P such that $\dim(D/P) = 1$, so that $(0) \subset Q \subset m$ is a saturated chain, while the longest saturated chains joining (0) to m have the form $(0) \subset P_1 \subset P_2 \subset m$. One has to work hard to construct Noetherian rings that are not catenary. Nagata also gives an example of a ring R that is catenary, but such that $R[x]$ is not catenary.

Notice that a localization or homomorphic image of a catenary ring is automatically catenary.

R is called *universally catenary* if every polynomial ring over R is catenary. This implies that every ring essentially of finite type over R is catenary.

A very important fact about Cohen-Macaulay rings is that they are catenary. Moreover, a polynomial ring over a Cohen-Macaulay ring is again a Cohen-Macaulay ring, which then implies that every Cohen-Macaulay ring is universally catenary. In particular, regular rings are universally catenary. Cohen-Macaulay local rings have a stronger property: they are

equidimensional, and all saturated chains from a minimal prime to the maximal ideal have length equal to the dimension of the local ring.

We shall prove the statements in the paragraph above. We first note:

Theorem. *If R is Cohen-Macaulay, so is the polynomial ring in n variables over R .*

Proof. By induction, we may assume that $n = 1$. Let \mathcal{M} be a maximal ideal of $R[X]$ lying over m in R . We may replace R by R_m and so we may assume that (R, m, K) is local. Then \mathcal{M} , which is a maximal ideal of $R[x]$ lying over m , corresponds to a maximal ideal of $K[x]$: each of these is generated by a monic irreducible polynomial f , which lifts to a monic polynomial F in $R[x]$. Thus, we may assume that $\mathcal{M} = mR[x] + FR[X]$. Let x_1, \dots, x_d be a system of parameters in R , which is also a regular sequence. We may kill the ideal generated by these elements, which also form a regular sequence in $R[X]_{\mathcal{M}}$. We are now in the case where R is an Artin local ring. It is clear that the height of \mathcal{M} is one. Because F is monic, it is not a zerodivisor: a monic polynomial over any ring is not a zerodivisor. This shows that the depth of \mathcal{M} is one, as needed. \square

Theorem. *Let (R, m, K) be a local ring and $M \neq 0$ a finitely generated Cohen-Macaulay R -module of Krull dimension d . Then every nonzero submodule N of M has Krull dimension d .*

Proof. We replace R by $R/\text{Ann}_R M$. Then every system of parameters for R is a regular sequence on M . We use induction on d . If $d = 0$ there is nothing to prove. Assume $d > 0$ and that the result holds for smaller d . If M has a submodule $N \neq 0$ of dimension $\leq d - 1$, we may choose N maximal with respect to this property. If N' is any nonzero submodule of M of dimension $< d$, then $N' \subseteq N$. To see this, note that $N \oplus N'$ has dimension $< d$, and maps onto $N + N' \subseteq M$, which therefore also has dimension $< d$. By the maximality of N , we must have $N + N' = N$. Since M is Cohen-Macaulay and $d \geq 1$, we can choose $x \in m$ not a zerodivisor on M , and, hence, also not a zerodivisor on N . We claim that x is not a zerodivisor on $\overline{M} = M/N$, for if $u \in M - N$ and $xu \in N$, then $Rxu \subseteq N$ has dimension $< d$. But this module is isomorphic with $Ru \subseteq M$, since x is not a zerodivisor, and so $\dim(Ru) < d$. But then $Ru \subseteq N$. Consequently, multiplication by x induces an isomorphism of the exact sequence $0 \rightarrow N \rightarrow M \rightarrow \overline{M} \rightarrow 0$ with the sequence $0 \rightarrow xN \rightarrow xM \rightarrow x\overline{M} \rightarrow 0$, and so this sequence is also exact. But we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & \overline{M} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & xN & \longrightarrow & xM & \longrightarrow & x\overline{M} & \longrightarrow & 0 \end{array}$$

where the vertical arrows are inclusions. By the nine lemma, or by an elementary diagram chase, the sequence of cokernels $0 \rightarrow N/xN \rightarrow M/xM \rightarrow \overline{M}/x\overline{M} \rightarrow 0$ is exact. Because x

is not a zerodivisor on M , it is part of a system of parameters for R , and can be extended to a system of parameters of length d , which is a regular sequence on M . Since x is a nonzerodivisor on N and M , $\dim(N/xN) = \dim(N) - 1 < d - 1$, while M/xM is Cohen-Macaulay of dimension $d - 1$. This contradicts the induction hypothesis. \square

Corollary. *If (R, m, K) is Cohen-Macaulay, R is equidimensional: every minimal prime \mathfrak{p} is such that $\dim(R/\mathfrak{p}) = \dim(R)$.*

Proof. If \mathfrak{p} is minimal, it is an associated prime of R , and we have $R/\mathfrak{p} \hookrightarrow R$. Since all nonzero submodules of R have dimension $\dim(R)$, the result follows. \square

Thus, a Cohen-Macaulay local ring cannot exhibit the kind of behavior one observes in $R = K[[x, y, z]]/((x, y) \cap (z))$: this ring has two minimal primes. One of them, \mathfrak{p}_1 , generated by the images of x and y , is such that R/\mathfrak{p}_1 has dimension 1. The other, \mathfrak{p}_2 , generated by the image of z , is such that R/\mathfrak{p}_2 has dimension 2. Note that while R is not equidimensional, it is still catenary.

We next observe:

Theorem. *In a Cohen-Macaulay ring R , if $P \subseteq Q$ are prime ideals of R then every saturated chain of prime ideals from P to Q has length $\text{height}(Q) - \text{height}(P)$. Thus, R is catenary.*

It follows that every ring essentially of finite type over a Cohen-Macaulay ring is universally catenary.

Proof. The issues are unaffected by localizing at Q . Thus, we may assume that R is local and that Q is the maximal ideal. There is part of a system of parameters of length $h = \text{height}(P)$ contained in P , call it x_1, \dots, x_h , by the Corollary near the bottom of p. 7 of the Lecture Notes of September 5. This sequence is a regular sequence on R and so on R_P , which implies that its image in R_P is system of parameters. We now replace R by $R/(x_1, \dots, x_h)$: when we kill part of a system of parameters in a Cohen-Macaulay ring, the image of the rest of that system of parameters is both a system of parameters and a regular sequence in the quotient. Thus, R remains Cohen-Macaulay. Q and P are replaced by their images, which have heights $\dim(R) - h$ and 0, and $\dim(R) - h = \dim(R/(x_1, \dots, x_h))$. We have therefore reduced to the case where (R, Q) is local and P is a minimal prime.

We know that $\dim(R) = \dim(R/P)$, and so at least one saturated chain from P to Q has length $\text{height}(Q) - \text{height}(P) = \text{height}(Q) - 0 = \dim(R)$. To complete the proof, it will suffice to show that all saturated chains from P to Q have the same length, and we may use induction on $\dim(R)$. Consider two such chains, and let their smallest elements other than P be P_1 and P'_1 . We claim that both of these are height one primes: if, say, P_1 is not height one we can localize at it and obtain a Cohen-Macaulay local ring (S, m) of dimension at least two and a saturated chain $\mathfrak{p} \subseteq m$ with $\mathfrak{p} = PS$ minimal in S . Choose an element $y \in m$ that is not in any minimal primes of S : its image will be a system of parameters for S/\mathfrak{p} , so that $Ry + \mathfrak{p}$ is m -primary. Extend y to a regular sequence of length

two in S : the second element has a power of the form $ry + u$, so that $y, ry + u$ is a regular sequence, and, hence, so is y, u . But then u, y is a regular sequence, a contradiction, since $u \in \mathfrak{p}$. Thus, P_1 (and, similarly, P'_1), have height one.

Choose an element f in P_1 not in any minimal prime of R , and an element g of P'_1 not in any minimal prime of R . Then fg is a nonzerodivisor in R , and P_1, P'_1 are both minimal primes of xy . The ring $R/(xy)$ is Cohen-Macaulay of dimension $\dim(R) - 1$. The result now follows from the induction hypothesis applied to $R/(xy)$: the images of the two saturated chains (omitting P from each) give saturated chains joining $P_1/(xy)$ (respectively, $P'_1/(xy)$) to $Q/(xy)$ in $R/(xy)$. These have the same length, and, hence, so did the original two chains.

The final statement now follows because a polynomial ring over a Cohen-Macaulay ring is again Cohen-Macaulay. \square

Excellent rings

A Noetherian ring R is called a *G-ring* (“G” as in “Grothendieck”) if for every local ring A of R , the map $A \rightarrow \widehat{A}$ is geometrically regular.

An *excellent* ring is a universally catenary Noetherian G-ring R such that in every finitely generated R -algebra S , the regular locus $\{P \in \text{Spec}(S) : S_P \text{ is regular}\}$ is Zariski open.

Excellent rings include the integers, fields, and complete local rings, as well as convergent power series rings over \mathbb{C} and \mathbb{R} . Every discrete valuation ring of equal characteristic 0 or of mixed characteristic is excellent. The following two results contain most of what we need to know about excellent rings.

Theorem. *Let R be an excellent ring. Then every localization of R , every homomorphic image of R , and every finitely generated R -algebra is excellent. Hence, every algebra essentially of finite type over R is excellent.*

Theorem. *Let R be an excellent ring.*

- (a) *If R is reduced, the normalization of R is module-finite over R .*
- (b) *If R is local and reduced, then \widehat{R} is reduced.*
- (c) *If R is local and equidimensional, then \widehat{R} is equidimensional.*
- (d) *If R is local and normal, then \widehat{R} is normal.*

For proofs of these results, we refer the reader to [H. Matsumura, *Commutative Algebra*, W.A. Benjamin, New York, 1970], as mentioned earlier.

Note that one does not expect the completion of an excellent local domain to be a domain. For example, consider the one-dimensional domain $S = \mathbb{C}[x, y]/(y^2 - x^2 - x^3)$. This is a domain because $x^2 + x^3$ is not a perfect square in $\mathbb{C}[x, y]$ (and, hence, not in its fraction field either, since $\mathbb{C}[x, y]$ is normal). If $m = (x, y)S$, then S_m is a local domain of dimension one. The completion of this ring is $\cong \mathbb{C}[[x, y]]/(y^2 - x^2 - x^3)$. This ring is not a domain: the point is that $x^2 + x^3 = x^2(1 + x)$ is a perfect square in the formal power series ring. Its square root may be written down explicitly using Newton's binomial theorem. Alternatively, one may see this using Hensel's Lemma: see p. 2 of the lecture notes of March 21 from Math 615, Winter 2007.

One does have from parts (b) and (c) of the Theorem above that the completion of an excellent local domain is reduced and equidimensional.

Example: a DVR that is not excellent. Let K be a perfect field of prime characteristic $p > 0$, and let

$$t_1, t_2, t_3, \dots, t_n, \dots$$

be countably many indeterminates over K . Let

$$L = K(t_1, \dots, t_n, \dots),$$

and let $L_n = L^p(t_1, \dots, t_n)$, which contains the p th power of every t_j and the first powers of t_1, \dots, t_n . Let x be a formal indeterminate, and let $V_n = L_n[[x]]$, a DVR in which every nonzero element is a unit times a power of x . Let

$$V = \bigcup_{n=1}^{\infty} V_n,$$

which is also a DVR in which every element is unit times a power of x . V has residue field L , and $\widehat{V} \cong L[[x]]$, but V only contains those power series such that all coefficients lie in a fixed choice of L_n . For example,

$$f = t_1x + t_2x^2 + \dots + t_nx^n + \dots \in \widehat{V} - V.$$

Note that the p th power of every element of \widehat{V} is in V . Thus, the generic fiber

$$\mathcal{K} = \text{frac}(V) \rightarrow \text{frac}(\widehat{V}) = \mathcal{L}$$

is a purely inseparable field extension, and is *not* geometrically regular. The ring

$$\mathcal{K}[f] \otimes_{\mathcal{K}} \mathcal{L}$$

is not even reduced: $f \otimes 1 - 1 \otimes f$ is a nonzero nilpotent. Thus, V is not a G-ring.
