Math 711: Lecture of September 24, 2007

Flat base change and Hom

We want to discuss in some detail when a short exact sequence splits. The following result is very useful.

Theorem (Hom commutes with flat base change). *If S is a flat R-algebra and M, N are R-modules such that M is finitely presented over R, then the canonical homomorphism*

$$\theta_M\colon S \otimes_R \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N)$$

sending $s \otimes f$ to $s(\mathbf{1}_S \otimes f)$ is an isomorphism.

Proof. It is easy to see that θ_R is an isomorphism and that $\theta_{M_1 \oplus M_2}$ may be identified with $\theta_{M_1} \oplus \theta_{M_2}$, so that θ_G is an isomorphism whenever G is a finitely generated free R-module.

Since M is finitely presented, we have an exact sequence $H \to G \twoheadrightarrow M \to 0$ where G, H are finitely generated free R-modules. In the diagram below the right column is obtained by first applying $S \otimes_R \_$ (exactness is preserved since $\otimes$ is right exact), and then applying $\mathrm{Hom}_S(\_, S \otimes_R N)$, so that the right column is exact. The left column is obtained by first applying $\mathrm{Hom}_R(\_, N)$, and then $S \otimes_R \_$ (exactness is preserved because of the hypothesis that S is R-flat). The squares are easily seen to commute.

$$\begin{array}{ccc}
S \otimes_R \mathrm{Hom}_R(H, N) & \xrightarrow{\theta_H} & \mathrm{Hom}_S(S \otimes_R H, S \otimes_R N) \\
\uparrow & & \uparrow \\
S \otimes_R \mathrm{Hom}_R(G, N) & \xrightarrow{\theta_G} & \mathrm{Hom}_S(S \otimes_R G, S \otimes_R N) \\
\uparrow & & \uparrow \\
S \otimes_R \mathrm{Hom}_R(M, N) & \xrightarrow{\theta_M} & \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N) \\
\uparrow & & \uparrow \\
0 & \longrightarrow & 0
\end{array}$$

From the fact, established in the first paragraph, that θ_G and θ_H are isomorphisms and the exactness of the two columns, it follows that θ_M is an isomorphism as well (kernels of isomorphic maps are isomorphic). $\square$

Corollary. *If W is a multiplicative system in R and M is finitely presented, we have that* $W^{-1}\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_{W^{-1}R}(W^{-1}M, W^{-1}N)$.

Moreover, if (R, m) is a local ring and both M, N are finitely generated, we may identify $\mathrm{Hom}_{\widehat{R}}(\widehat{M}, \widehat{N})$ *with the m-adic completion of* $\mathrm{Hom}_R(M, N)$ *(since m-adic completion is the same as tensoring over R with $\widehat{R}$ (as covariant functors) on finitely generated R-modules).* $\square$

When does a short exact sequence split?

Throughout this section, $0 \longrightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \to 0$ is a short exact sequence of modules over a ring R. There is no restriction on the characteristic of R. We want to discuss the problem of when this sequence splits. One condition is that there exist a map $\eta : M \to N$ such that $\eta\alpha = \mathbf{1}_N$. Let $Q' = \mathrm{Ker}\,(\eta)$. Then Q' is disjoint from the image $\alpha(N) = N'$ of N in M, and $N' + Q' = M$. It follows that M is the internal direct sum of N' and Q' and that β maps Q' isomorphically onto Q.

Similarly, the sequence splits if there is a map $\theta : Q \to M$ such that $\beta\theta = \mathbf{1}_Q$. In this case let $N' = \alpha(N)$ and $Q' = \theta(Q)$. Again, N' and Q' are disjoint, and $N' + Q' = M$, so that M is again the internal direct sum of N' and Q'.

Proposition. *Let R be an arbitrary ring and let*

$$(\#) \quad 0 \longrightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \to 0$$

be a short exact sequence of R-modules. Consider the sequence

$$(*) \quad 0 \longrightarrow \mathrm{Hom}_R(Q,\, N) \xrightarrow{\alpha_*} \mathrm{Hom}_R(Q,\, M) \xrightarrow{\beta_*} \mathrm{Hom}_R(Q,\, Q) \to 0$$

which is exact except possibly at $\mathrm{Hom}_R(Q,\, Q)$*, and let* $C = \mathrm{Coker}\,(\beta_*)$*. The following conditions are equivalent:*

(1) *The sequence $(\#)$ is split.*

(2) *The sequence $(*)$ is exact.*

(3) *The map β_* is surjective.*

(4) $C = 0$.

(5) *The element $\mathbf{1}_Q$ is in the image of β_*.*

Proof. Because Hom commutes with finite direct sum, we have that (1) $\Rightarrow$ (2), while (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) is clear. It remains to show that (5) $\Rightarrow$ (1). Suppose $\theta : Q \to M$ is such that $\beta_*(\theta) = \mathbf{1}_Q$. Since β_* is induced by composition with β, we have that $\beta\theta = \mathbf{1}_Q$. $\square$

A split exact sequence remains split after any base change. In particular, it remains split after localization. There are partial converses. Recall that if $I \subseteq R$,

$$\mathcal{V}(I) = \{P \in \mathrm{Spec}\,(R) : I \subseteq P\},$$

and that

$$\mathcal{D}(I) = \mathrm{Spec}\,(R) - \mathcal{V}(I).$$

In particular,

$$\mathcal{D}(fR) = \{P \in \mathrm{Spec}\,(R) : f \notin P\},$$

and we also write $\mathcal{D}(f)$ or $\mathcal{D}_f$ for $\mathcal{D}(fR)$.

Theorem. *Let R be an arbitrary ring and let*

$$(\#) \quad 0 \longrightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \longrightarrow 0$$

be a short exact sequence of R-modules such that Q is finitely presented.

(a) *(#) is split if and only if for every maximal ideal m of R, the sequence*

$$0 \to N_m \to M_m \to Q_m \to 0$$

is split.

(b) *Let S be a faithfully flat R-algebra. The sequence (#) is split if and only if the sequence*

$$0 \to S \otimes_R N \to S \otimes_R M \to S \otimes_R Q \to 0$$

is split.

(c) *Let W be a multiplicative system in R. If the sequence*

$$0 \to W^{-1}N \to W^{-1}M \to W^{-1}Q \to 0$$

is split over $W^{-1}R$, then there exists a single element $c \in W$ such that

$$0 \to N_c \to M_c \to Q_c \to 0$$

is split over R_c.

(d) *If P is a prime ideal of R such that*

$$0 \to N_P \to M_P \to Q_P \to 0$$

is split, there exists an element $c \in R - P$ such that

$$0 \to N_c \to M_c \to Q_c \to 0$$

is split over R_c. Hence, (#) becomes split after localization at any prime P' that does not contain c, i.e., any prime P' such that $c \notin P'$.

(e) *The split locus for (#), by which we mean the set of primes $P \in \mathrm{Spec}\,(R)$ such that*

$$0 \to N_P \to M_P \to Q_P \to 0$$

is split over R_P, *is a Zariski open set in* Spec$\,(R)$.

Proof. Let $C = \mathrm{Coker}\big(\mathrm{Hom}(Q,\, M) \to \mathrm{Hom}_R(Q,\, Q)\big)$, as in the preceding Proposition, and let γ denote the image of $\mathbf{1}_Q$ in C. By part (4) of the preceding Propostion, (#) is split if and only if $\gamma = 0$.

(a) The "only if" part is clear, since splitting is preserved by any base change. For the "if" part, suppose that $\gamma \neq 0$. The we can choose a maximal ideal m in the support of $R\gamma \subseteq C$, i.e., such that $\mathrm{Ann}_R\gamma \subseteq m$. The fact that Q is finitely presented implies that localization commutes with Hom. Thus, localizing at m yields

$$0 \to \mathrm{Hom}_{R_m}(Q_m,\, N_m) \to \mathrm{Hom}_{R_m}(Q_m,\, M_m) \to \mathrm{Hom}_{R_m}(Q_m,\, Q_m) \to C_m \to 0,$$

and since the image of γ is not 0, the sequence $0 \to N_m \to M_m \to Q_m \to 0$ does not split.

(b) Again, the "only if" part is clear, and since Q is finitely presented and S is flat, Hom commutes with base change to S. After base change, the new cokernel is $S \otimes_R C$. But $C = 0$ if and only if $S \otimes_R C = 0$, since S is faithfully flat, and the result follows.

(c) Similarly, the sequence is split after localization at W if and only if the image of γ is 0 after localization at W, and this happens if and only if $c\gamma = 0$ for some $c \in W$. But then localiziing at the element c kills γ.

(d) This is simply part (c) applied with $W = R - P$

(e) If P is in the split locus and $c \notin P$ is chosen as in part (d), $\mathcal{D}(c)$ is a Zariski open neighborhood of P in the split locus. $\square$

Behavior of strongly F-regular rings

Theorem. *Let R be an F-finite reduced ring. Then the following conditions are equivalent:*

(1) *R is strongly F-regular.*

(2) *R_m is strongly F-regular for every maximal ideal m of R.*

(3) *$W^{-1}R$ is strongly F-regular for every multiplicative system W in R.*

Proof. We shall show that (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

To show that (1) $\Rightarrow$ (3), suppose that R is strongly F-regular and let W be a multiplicative system. By the Proposition on p. 2 of the Lecture Notes of September 17, every element of $(W^{-1}R)^\circ$ has the form c/w where $w \in W$ and $c \in R^\circ$. Given such an element c/w, we can choose q_c and an R-linear map $\theta : R^{1/q_c} \to R$ such that $\theta(c^{1/q}) = 1$. After localization at c, θ induces a map $\theta_c : (R_c)^{1/q_c} \to R_c$ sending $(c/1)^{1/q_c}$ to $1/1$. Define $\eta : (R_c)^{1/q_c} \to R_c$ by $\eta(u) = \theta_c(w^{1/q_c}u)$. Then $\eta : (R_c)^{1/q_c} \to R_c$ is an R_c-linear map such that $\eta\big((c/w)^{1/q_c}\big) = 1$, as required. (3) $\Rightarrow$ (2) is obvious.

It remains to show that (2) $\Rightarrow$ (1). Fix $c \in R^\circ$. Then for every maximal ideal m of R, the image of c is in $(R_m)^\circ$, and so there exist q_m and a splitting of the map $R_m \to$

$(R^{1/q_m})m \cong (R_m)^{1/q_m}$ that sends $1 \mapsto c^{1/q_m}$. Then there is also such a splitting of the map $R \to R^{1/q_m}$ after localizing at any prime in a Zariski neighborhood U_m of m. Since the U_m cover MaxSpec (R), they cover Spec (R), and by the quasicompactness of Spec (R) there are finitely many maximal ideals $m_1, \ldots, m_n$ such that the open sets $U_{m_1}, \ldots, U_{m_n}$ cover Spec (R). Let $q_c = \max\{q_{m_1}, \ldots, q_{m_n}\}$. Then the map $R \to R^{1/q_c}$ that sends $1 \mapsto c^{1/q_c}$ splits after localizing at any maximal ideal in any of the U_{m_i}, i.e., after localizing at any maximal ideal. By part (a) of the preceding Proposition, the map $R \to R^{1/q_c}$ sending $1 \mapsto c^{1/q_c}$ splits, as required. □

Corollary. *A strongly F-regular ring is F-regular.*

Proof. This is immediate from the fact that strongly F-regular rings are weakly F-regular and the fact that a localization of a strongly F-regular ring is strongly F-regular. □

Corollary. *R is strongly F-regular if and only if it is a finite product of strongly F-regular domains.*

Proof. If R is strongly F-regular it is normal, and, therefore, a product of domains. Since the issue of whether R is strongly F-regular is local on the maximal ideals of R, when R is a product of domains it is strongly F-regular if and only if each of the factor domains is strongly F-regular. □

Proposition. *If S is strongly F-regular and R is a direct summand of S, then R is strongly F-regular.*

Proof. If R and S are domains, we may proceed as follows. Let $c \in R^\circ = R - \{0\}$ be given. Since S is strongly F-regular we may choose q and an S-linear map $\theta : S^{1/q} \to S$ such that $\theta(c^{1/q}) = 1$. Let $\alpha : S \to R$ be R-linear such that $\alpha(1) = 1$. Then $\alpha \circ \theta : S^{1/q} \to R$ is R-linear and sends $c^{1/q} \mapsto 1$. We may restrict this map to $R^{1/q}$.

In the general case, we may first localize at a prime of R: it suffices to see that every such localization is strongly F-regular. S is a product of F-regular domains $S_1 \times \cdots \times S_n$ each of which is an R-algebra. Let $\alpha : S \to R$ be such that $\alpha(1) = 1$. The element $1 \in S$ is the sum of n idempotents e_i, where e_i has component 0 in S_j for $j \neq i$ while the component in S_i is 1. Then $1 = \alpha(1) = \sum_{i=1}^n \alpha(e_i)$, and since R is local, at least one $\alpha(e_i)$ is not in the maximal ideal m of the local ring R, i.e., we can fix i such that $\alpha(e_i)$ is a unit a of R. We have an R-linear injection $\iota : S_i \to S$ by identifying S_i with $0 \times 0 \times S_i \times 0 \times 0$, i.e., with the set of elements of S all of whose coordinates except the ith are 0. Then $a^{-1}\alpha \circ \iota$ is a splitting of $R \to S_i$ over R, and so we have reduced to the domain case, which was handled in the first paragraph. □

We also have:

Proposition. *If $R \to S$ is faithfully flat and S is strongly F-regular then R is strongly F-regular.*

Proof. Let $c \in R^\circ$. Then $c \in S^\circ$, and so there exists q and an S-linear map $S^{1/q} \to S$ such that $c^{1/q} \mapsto 1$. There is an obvious map $S \otimes_R R^{1/q} \to S^{1/q}$, since both factors in the tensor product have maps to $S^{1/q}$ as R-algebras. This yields a map $S \otimes_R R^{1/q} \to S = S \otimes_R R$ that sends $1 \otimes c^{1/q} \mapsto 1 \otimes 1$ that is S-linear. This implies that the map $R \to R^{1/q}$ sending $1 \mapsto c^{1/q}$ splits after a faithfully flat base change to S. By part (b) of the Theorem at the top of p. 3, the map $R \to R^{1/q}$ such that $1 \mapsto c^{1/q}$ spits over R, as required. □

Theorem. *An F-finite regular ring is strongly F-regular.*

Proof. We may assume that $(R,\, m,\, K)$ is local: it is therefore a domain. Let $c \neq 0$ be given. Choose q so large that $c \notin m^{[q]}$: this is possible because $\bigcap_q m^{[q]} \subseteq \bigcap_q m^q = (0)$. The flatness of Frobenius implies that $R^{1/q}$ is flat and, therefore, free over R since $R^{1/q}$ is module-finite over R. (Alternatively, $R^{1/q}$ is free because a regular system of parameters $x_1, \, \ldots \,, x_n$ in R is a regular sequence on $R^{1/q}$: one may apply the Lemma in the middle of p. 8 of the Lecture Notes of September 5. Or one may further reduce to the case where R is complete, using the preceding Proposition, and even pass from the complete regular ring $K[[x_1, \, \ldots \,, x_n]$ to $\overline{K}[[x_1, \, \ldots \,, x_n]]$, where $\overline{K}$ is an algebraic closure of K. This is simply a further faithfully flat extension. Now the fact that $R^{1/q}$ is R-free is easy.) Since $c \notin m^{[q]}$, we have that $c^{1/q} \notin mR^{1/q}$. By Nakayama's Lemma, $c^{1/q}$ is part of a minimal basis for the R-free module $R^{1/q}$, and a minimal basis is a free basis. It follows that there is an R-linear map $R^{1/q} \to R$ such that $c^{1/q} \mapsto 1$: the values can be specified arbitrarily on a free basis containing $c^{1/q}$. □

Remark: qth roots of maps. The following situation arises frequently in studying strongly F-regular rings. One has q, q_0, q_1, q_2, where these are all powers of p, the prime characteristic, such that $q_0 \leq q_1$ and $q_0 \leq q_2$, and we have an R^{1/q_0}-linear map $\alpha : R^{1/q_1} \to R^{1/q_2}$. This map might have certain specified values, e.g., $\alpha(u) = v$. Here, one or more of the integers q, q_i may be 1. Then one has a map which we denote $\alpha^{1/q} : R^{1/q_1 q} \to R^{1/q_2 q}$ which is $R^{1/q_0 q}$-linear, that is simply defined by the rule $\alpha^{1/q}(s^{1/q}) = \alpha(s)^{1/q}$. Then $\alpha^{1/q}(u^{1/q}) = v^{1/q}$.

The following result makes the property of being a strongly F-regular ring much easier to test: instead of needing to worry about constructing a splitting for every element of R°, one only needs to construct a splitting for *one* element of R°.

Theorem. *Let R be a reduced F-finite ring of prime characteristic $p > 0$, and let $c \in R^\circ$ be such that R_c is strongly F-regular. Then R is strongly F-regular if and only if*

(∗) *there exists q_c such that the map $R \to R^{1/q_c}$ sending $1 \mapsto c^{1/q_c}$ splits.*

Proof. The condition (∗) is obviously necessary for R to be strongly F-regular: we need to show that it is sufficient. Therefore, assume that we have an R-linear splitting

$$\theta : R^{1/q_c} \to R,$$

with $\theta(c^{1/q_c}) = 1$. By the Remark beginning near the bottom of p. 4 of the Lecture Notes of September 21, we know that R is F-split. Suppose that $d \in R^\circ$ is given.

Since R_c is strongly F-regular we can choose q_d and an R_c-linear map $\beta : R_c^{1/q_d} \to R_c$ such that $\beta(d^{1/q_d}) = 1$. Since $\operatorname{Hom}_{R_c}(R_c^{1/q_d}, R_c)$ is the localization of $\operatorname{Hom}_R(R^{1/q_d}, R)$ at c, we have that $\beta = \dfrac{1}{c^q}\alpha$ for some sufficiently large choice of q: since we are free to make the power of c in the denominator larger if we choose, there is no loss of generality in assuming that the exponent is a power of p. Then $\alpha : R^{1/q_d} \to R$ is an R-linear map such that

$$\alpha(d^{1/q_d}) = c^q \beta(d^{1/q_d}) = c^q.$$

By taking qq_c roots we obtain a map

$$\alpha^{1/qq_c} : R^{1/qq_cq_d} \to R^{1/qq_c}$$

that is R^{1/qq_c}-linear and sends $d^{1/qq_cq_d} \mapsto c^{1/q_c}$. Because R is F-split, the inclusion $R \hookrightarrow R^{1/q}$ splits: let $\gamma : R^{1/q} \to R$ be R linear such that $\gamma(1) = 1$. Then $\gamma^{1/q_c} : R^{1/qq_c} \to R^{1/q_c}$ is an R^{1/q_c}-linear retraction and sends $c^{1/q_c} \mapsto c^{1/q_c}$. Then $\theta \circ \gamma^{1/q_c} \circ \alpha^{1/qq_c} : R^{1/qq_cq_d} \to R$ and sends $d^{1/qq_cq_d} \mapsto 1$, as required. $\square$