Math 711: Lecture of September 26, 2007

We want to use the theory of strongly F-regular F-finite rings to prove the existence of test elements.

We first prove two preliminary results:

Lemma. *Let R be an F-finite reduced ring and $c \in R^\circ$ be such that R_c is F-split (which is automatic if R_c is strongly F-regular). Then there exists an R-linear map $\theta : R^{1/p} \to R$ such that the value on 1 is a power of c.*

Proof. We can choose an R_c-linear map $(R_c)^{1/p} \to R_c$ such that $1 \mapsto 1$, and

$$(R_c)^{1/p} \cong (R^{1/p})_c.$$

Then $\mathrm{Hom}_{R_c}(R_c^{1/p},\, R_c)$ is the localization of $\mathrm{Hom}_R(R^{1/p},\, R)$ at c, and so we can write $\theta = \dfrac{1}{c^N}\alpha$, where $N \in \mathbb{N}$ and $\alpha : R^{1/p} \to R$ is R-linear. But then $\alpha = c^N\beta$ and so $\alpha(1) = c^N\beta(1) = c^N$, as required. $\square$

Lemma. *Let R be a reduced F-finite ring and suppose that there exists an R-linear map $\theta : R^{1/p} \to R$ such that $\theta(1) = c \in R^\circ$. Then for every $q = p^e$, there exists an R-linear map $\eta_q : R^{1/q} \to R$ such that $\eta_q(1) = c^2$.*

Proof. We use induction on q. If $q = 1$ we may take $\eta_1 = c^2\mathbf{1}_R$, and if $q = p$ we may take $\eta_p = c\,\theta$. Now suppose that η_q has been constructed for $q \geq p$. Then $\eta_q^{1/p} : R^{1/pq} \to R^{1/p}$, it is $R^{1/p}$-linear, hence, R-linear, and its value on 1 is $c^{2/p}$. Define

$$\eta_{pq}(u) = \theta\bigl(c^{(p-2)/p}\eta_q(u)\bigr).$$

Consequently, we have, as required, that

$$\eta_{pq}(1) = \theta\bigl(c^{(p-2)/p}\eta_q(1)\bigr) = \theta(c^{(p-2)/p}c^{2/p}) = \theta(c) = c\theta(1) = c^2. \qquad \square$$

We can now prove the following:

Theorem (existence of big test elements). *Let R be F-finite and reduced. If $c \in R^\circ$ and R_c is strongly F-regular, then c has a power that is a big test element. If R_c is strongly F-regular and there exists an R-linear map $\theta : R^{1/p} \to R$ such that $\theta(1) = c$, then c^3 is a big test element.*

Proof. Since R_c is strongly F-regular it is F-split. By the first Lemma on p. 1 there exist an integer N and an R-linear map $\theta : R^{1/p} \to R$ such that $\theta(1) = c^N$. By the second statement of the Theorem, c^{3N} is then a big test element, and so it suffices to prove the second statement.

Suppose that c satsifies the hypothesis of the second statement. By part (a) of the Proposition at the bottom of p. 8 of the Lecture Notes of September 17, it suffices to show that if $N \subseteq M$ are arbitrary modules and $u \in N^*_M$, then $c^3 u \in N$. We may map a free module G onto M, let H be the inverse image of N in G, and let $v \in G$ be an element that maps to $u \in N$. Then we have $v \in H^*_G$, and it suffices to prove that $c^3 v \in H$. Since $v \in H^*_G$ there exists $d \in R^\circ$ such that $dv^q \in H^{[q]}$ for all $q \geq q_1$. Since R_c is strongly F-regular, there exist q_d and an R_c-linear map $\beta : (R_c)^{1/q_d} \to R_c$ that sends $d^{1/q_d} \to 1$: we may take q_d larger, if necessary, and so we may assume that $q_d \geq q_1$. As usual, we may assume that $\beta = \dfrac{1}{c^q}\alpha$ where $\alpha : R^{1/q_d} \to R$ is R-linear. Hence, $\alpha = c^q \beta$, and $\alpha(d^{1/q_d}) = c^q$. It follows that $\alpha^{1/q} : R^{1/q_d q} \to R^{1/q}$ is $R^{1/q}$-linear, hence, R-linear, and its value on 1 is c. By the preceding Lemma we have an R-linear map $\eta_q : R^{1/q} \to R$ whose value on 1 is c^2, so that $\eta_q(c) = c\eta^q(1) = c^3$. Let $\gamma = \eta_q \circ \alpha^{1/q}$, which is an R-linear map $R^{1/q_d q} \to R$ sending $d^{1/q_d q}$ to $\eta_q(c) = c^3$. Since $q_d q \geq q_1$, we have $dv^{q_d q} \in H^{[q_d q]}$, i.e.,

$$(\#) \quad dv^{q_d q} = \sum_{i=1}^{n} r_i h_i^q$$

for some integer $n > 0$ and elements $r_1, \ldots, r_n \in R$ and $h_1, \ldots, h_n \in H$.

Consider $G' = R^{1/qq_d} \otimes G$. We identify G with its image under the map $G \to G'$ that sends $g \mapsto 1 \otimes g$. Thus, if $s \in R^{1/q_d q}$, we may write sg instead of $s \otimes g$. Note that G' is free over R^{1/qq^d}, and the R-linear map $\gamma : R^{1/qq_d} \to R$ induces an R-linear map

$$\gamma' : G' = R^{qq_d} \otimes_R G \to R \otimes_R G \cong G$$

that sends $sg \mapsto \gamma(s)g$ for all $s \in R^{1/qq_d}$ and all $g \in G$. Note that by taking $q_d q$ th roots in the displayed equation (#) above, we obtain

$$(\dagger) \quad d^{1/q_d q} v = \sum_{i=1}^{n} r_i^{1/q_d q} h_i.$$

We may now apply γ' to both sides of (†): we have

$$c^3 v = \sum_{i=1}^{n} \gamma(r_i^{1/q_d q}) h_i \in H,$$

exactly as required. $\square$

Disccussion. As noted on the bottom of p. 2 and top of p. 3 of the Lecture Notes of September 21, it follows that every F-finite reduced ring has a big test element: one can choose $c \in R^\circ$ such that R_c is regular. This is a consequence of the fact that F-finite rings are excellent. But one can give a proof of the existence of such elements c in F-finite rings of characteristic p very easily if one assumes that a Noetherian ring is regular if and only if the Frobenius endomorphism is flat (we proved the "only if" direction earlier). See [E. Kunz, *Characterizations of regular local rings of characteristic p,* Amer. J. Math. **91** (1969) 772–784]. Assuming the "if" direction, we may argue as follows. First note that one can localize at one such element c so that the idempotent elements of the total quotient ring of R are in the localization. Therefore, there is no loss of generality in assuming that R is a domain. Then $R^{1/p}$ is a finitely generated torsion-free R-module. Choose a maximal set $s_1, \dots, s_n$ of R-linearly independent elements in $R^{1/p}$. This gives an inclusion

$$R^n \cong Rs_1 + \cdots + Rs_n \subseteq R^{1/p}.$$

Call the cokernel C. Then C is finitely generated, and C must be a torsion module over R: if $s_{n+1} \in R^{1/p}$ represents an element of C that is not a torsion element, then $s_1, \dots, s_{n+1}$ are linearly independent over R, a contradiction. Hence, there exists $c \in R^\circ$ that kills C, and so $cR^{1/p} \subseteq R^n$. It follows that $(R^{1/p})_c \cong R_c^n$, and so $(R_c)^{1p}$ is free over R_c. But this implies that F_{R_c} is flat, and so R_c is regular, as required. $\square$

In any case, we have proved:

Corollary. *If R is reduced and F-finite, then R has a big test element. Hence, $\tau_{\mathrm{b}}(R)$ is generated by the big test elements of R, and $\tau(R)$ is generated by the test elements of R.* $\square$

Our next objective is to show that the big test elements produced by the Theorem on p. 1 are actually completely stable. In fact, we shall prove something more: they remain test elements after any geometrically regular base change, i.e., their images under a flat map $R \to S$ with geometrically regular fibers are again test elements.