Math 711: Lecture of September 28, 2007

We next want to note some elementary connections between properties of regular sequences and the vanishing of Tor.

Proposition. *Let $x_1, \ldots, x_n \in R$ and let M be an R-module. Suppose that $x_1, \ldots, x_n$ is a possibly improper regular sequence in R, and is also a possibly improper regular sequence on M. Let $I_k = (x_1, \ldots, x_k)R$, $0 \leq k \leq n$, so that $I_0 = 0$. Then*

$$\mathrm{Tor}_i^R(R/I_k, M) = 0$$

for $i \geq 1$ and $0 \leq k \leq n$.

Proof. If $k = 0$ this is clear, since R is free and has a projective resolution in which the terms with a positive index all vanish. We use induction on k. We assume the result for some $k < n$, and we prove it for $k+1$. From the short exact sequence

$$0 \to R/I_k \xrightarrow{x_{k+1}\cdot} R/I_k \to R/I_{k+1} \to 0$$

we have a long exact sequence for Tor, part of which is

$$\mathrm{Tor}_i^R(R/I_k,\, M) \to \mathrm{Tor}_i^R(R/I_{k+1},\, M) \to \mathrm{Tor}_{i-1}^R(R/I_k,\, M) \xrightarrow{x_{k+1}\cdot} \mathrm{Tor}_{i-1}^R(R/I_k,\, M)$$

If $i \geq 2$, the result is immediate from the induction hypothesis, because the terms surrounding $\mathrm{Tor}_i^R(R/I_{k+1},\, M)$ are 0. If $i = 1$, this becomes:

$$0 \to \mathrm{Tor}_1^R(I_{k+1},\, M) \to M/I_kM \xrightarrow{x_{k+1}\cdot} M/I_kM$$

which shows that $\mathrm{Tor}_1^R(I_{k+1},\, M)$ is isomorphic with the kernel of the map given by multiplication by x_{k+1} on M/I_kM, and this is 0 because $x_1, \ldots, x_n$ is a possibly improper regular sequence on M. $\square$

The following result was stated earlier, in the Lecture Notes of September 14, p. 2. where it was used to give one of the proofs of the flatness of the Frobenius endomorphism for a regular ring R. We now give a proof, but in the course of the proof, we assume the theorem that $(*)$ over a regular local ring, every module has a finite free resolution of length at most the dimension of the ring. The condition $(*)$ actually characterizes regularity. Later, we shall develop results on Koszul complexes that permit a very easy proof that the condition $(*)$ holds over every regular local ring.

Theorem. *Let (R, m, K) be a regular local ring and let M be an R-module. Them M is a big Cohen-Macaulay module over R if and only if M is faithfully flat over R.*

Proof. By the comments at the bottom of p. 2 and top of p. 3 of the Lecture Notes of September 14, we already know that both conditions in the Theorem imply that $mM \neq M$, and that a faithfully flat module is a big Cohen-Macaulay module. It remains only to prove that if M is a big Cohen-Macaulay module over R, then M is flat.

It suffices to show that for every R-module N and every $i \geq 1$, $\mathrm{Tor}_i^R(N, M) = 0$. In fact, it suffices to show this when $i = 1$, for then if $0 \to N_0 \to N_1 \to N \to 0$ is exact, we have

$$0 = \mathrm{Tor}_1^R(M, N) \to M \otimes_R N_0 \to M \otimes_R N_1$$

is exact, which yields the needed injectivity. However, we carry through the proof by reverse induction on i, so that we need to consider all $i \geq 1$.

Because Tor commutes with direct limits, we may reduce to the case where N is finitely generated. We then know that $\mathrm{Tor}_i^R(N, M) = 0$ for $i > n$, because N has a free resolution of length at most n by the condition $(*)$ satisfied by regular local rings that was discussed in the paragraph just before the statement of the Theorem. Hence, it suffices to prove that if $i \geq 1$ and for all finitely generated R-modules N we have that $\mathrm{Tor}_j^R(N, M) = 0$ for $j \geq i+1$, then $\mathrm{Tor}_i(N, M) = 0$ for all finitely generated R-modules N as well.

We first consider the case where $N = R/P$ is prime cyclic. By the Corollary near the bottom of p. 7 of the Lecture Notes of September 5, there is a regular sequence whose length is the height of P contained in P, say $x_1, \ldots, x_h$, and then P is a minimal prime of $R/(x_1, \ldots, x_h)$. This implies that P is also an assoicated prime of $R/(x_1, \ldots, x_h)R$, so that we have a short exact sequence

$$0 \to R/P \to R/(x_1, \ldots, x_h)R \to C \to 0$$

for some R-module C. The long exact sequence for Tor then yields, in part:

$$\mathrm{Tor}_{i+1}^R(C, M) \to \mathrm{Tor}_i^R(R/P, M) \to \mathrm{Tor}_i^R(R/(x_1, \ldots, x_h)R, M)$$

The leftmost term is 0 by the induction hypothesis, and the rightmost term is 0 by the first Proposition on p. 1. Hence, $\mathrm{Tor}_i^R(R/P, M) = 0$.

We can now proceed by induction on the least number of factors in a finite filtration of N by prime cyclic modules. The case where there is just one factor was handled in the preceding paragraph. Suppose that $R/P = N_1 \subseteq N$ begins such a filtration. Then N/N_1 has a shorter filtration. The long exact sequence for Tor yields

$$\mathrm{Tor}_i^R(R/P, M) \to \mathrm{Tor}_i^R(N, M) \to \mathrm{Tor}_i^R(N/N_1, M).$$

The first term vanishes by the result of the preceding paragraph, and the third term by the induction hypothesis. □

We are aiming to prove that if $R \to S$ is geometrically regular (i.e., flat, with geometrically regular fibers) and R is strongly F-regular, then S is strongly F-regular. In order to prove this, we will make use of the following result:

Theorem (Radu-André). *Let $R \to S$ be a geometrically regular map of F-finite rings of prime characteristic $p > 0$. Then for all q, the map $R^{1/q} \otimes_R S \to S^{1/q}$ is faithfully flat.*

The Radu-André theorem asserts the same conclusion even when R and S are not assumed to be F-finite. In fact, $R \to S$ is geometrically regular if and only if the homomorphisms $R^{1/q} \otimes_R S \to S^{1/q}$ are flat. However, we do not need the converse, and we only need the theorem in the F-finite case, where the argument is easier.

The proof of this Theorem will require some effort. We first want to note that it has the following consequence:

Corollary. *Let $R \to S$ be a geometrically regular map of F-finite rings of prime characteristic $p > 0$. Then for all q, the map $R^{1/q} \otimes_R S \to S^{1/q}$ makes $R^{1/q} \otimes_R S$ a direct summand of $S^{1/q}$.*

Note that since $S \to S^{1/q}$ is module-finite, we have that $R^{1/q} \otimes_R S \to S^{1/q}$ is module-finite as well. The Corollary above then follows from the Radu-André Theorem and the following fact.

Proposition. *Let $A \to B$ be a faithfully flat map of Noetherian rings such that B is module-finite over A. Then A is a direct summand of B as an A-module.*

Proof. The issue is local on A. But when (A, m, K) is local, a finitely generated module is flat if and only if it is free, and so B is a nonzero free A-algebra. The element $1 \in B$ is not in mB, and so is part of a minimal basis, which will be a free basis, for B over A. Hence, there is an A-linear map $B \to A$ whose value on $1 \in B$ is $1 \in A$. $\square$