Math 711: Lecture of October 1, 2007

In the proof of the Radu-André Theorem we will need the result just below. A more general theorem may be found in [H. Matsumura, *Commutative Algebra*, W. A. Benjamin, New York, 1 970], Ch. 8 (20.C) Theorem 49, p. 146, but the version we give here will suffice for our purposes.

First note the following fact: if $I \subseteq A$ is an ideal and M is an A-module, then $\mathrm{Tor}_1^A(A/I,\, M) = 0$ if and only if the map $I \otimes_A M \to IM$, which is alway surjective, is an isomorphism. This map sends $i \otimes u \mapsto iu$. The reason is that we may start with the short exact sequence $0 \to I \to A \to A/I \to 0$ and apply $\_ \otimes_A M$. The long exact sequence then gives, in part:

$$0 = \mathrm{Tor}_1^A(A,\, M) \to \mathrm{Tor}_1^A(A/I,\, M) \to I \otimes_A M \to M$$

The image of the rightmost map is IM, and so we have

$$0 \to \mathrm{Tor}_1^A(A/I,\, M) \to I \otimes_A M \to IM \to 0$$

is exact, from which the statement we want is clear.

Theorem (local criterion for flatness). *Let $A \to B$ be a local homomorphism of local rings, let M be a finitely generated B-module and let I be a proper ideal of A. Then the following three conditions are equivalent:*

(1) *M is flat over A.*

(2) *M/IM is flat over A/I and $I \otimes_A M \to IM$ is an isomorphism.*

(3) *M/IM is flat over A/I and $\mathrm{Tor}_1^A(A/I,\, M) = 0$.*

Proof. The discussion of the preceding paragraph shows that (2) $\Leftrightarrow$ (3), and (1) $\Rightarrow$ (3) is clear. It remains to prove (3) $\Rightarrow$ (1), and so we assume (3). To show that M is flat, it suffices to show that if $N_0 \subseteq N$ is an injection of finitely generated R-modules then $N_0 \otimes_A M \to N \otimes_A M$ is injective. Moreover, by the Proposition at the bottom of p. 3 of the Lecture Notes of September 14, we need only prove this when N has finite length. Consequently, we may assume that N is killed by a power of I, and so we have that $I^k N \subseteq N_0$ for some k. Let $N_i = N_0 + I^{k-i}N$ for $0 \leq i \leq k$. The $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_k = N$, and it suffices to show that $N_j \otimes_A M \to N_{j+1} \otimes_A M$ is injective for each j. We have now reduced to the case where $Q = N_{j+1}/N_j$ is killed by I. From the long exact sequence for Tor arising from applying $\_ \otimes_A M$ to the short exact sequence

$$0 \to N_j \to N_{j+1} \to Q \to 0,$$

we have

$$\mathrm{Tor}_1^A(Q,\, M) \to N_j \otimes_A M \to N_{j+1} \otimes_A M$$

is exact, and so it suffices to show that if Q is a finitely generated A-module killed by I, then $\mathrm{Tor}_1^A(Q,\, M) = 0$.

Since Q is killed by I, we may think of it as a finitely generated module over A/I. Hence, there is a short exact sequence

$$0 \to Z \to (A/I)^{\oplus h} \to Q \to 0.$$

Applying $\_ \otimes_A M$, from the long exact sequence for Tor we have that

$$\mathrm{Tor}_1^A\big((A/I)^{\oplus h},\, M\big) \to \mathrm{Tor}_1^A(Q,\, M) \to Z \otimes_A M \xrightarrow{\alpha} (A/I)^{\oplus h} \otimes_A M$$

is exact. By hypothesis, $\mathrm{Tor}_1^A(A/I,\, M) = 0$, and so the leftmost term is 0. It follows that $\mathrm{Tor}_1^A(Q,\, M) \cong \mathrm{Ker}\,(\alpha)$. To conclude the proof, it will suffice to show that α is injective.

Hence, it is enough to show that $\_ \otimes_A M$ is an exact functor on A-modules Y that are killed by I. For such an A-module Y we have that

$$Y \otimes_A M \cong (Y \otimes_{A/I} A/I) \otimes_A M \cong Y \otimes_{A/I} \big((A/I) \otimes_A M\big) \cong Y \otimes_{A/I} M/IM,$$

and this is an isomorphism as functors of Y. Since M/IM is flat over A/I, the injectivity of α follows. $\square$

We want to record the following observation.

Proposition. *Let $f : A \to B$ be a homorphism of Noetherian rings of prime characteristic $p > 0$ such that the kernel of f consists of nilpotent elements of A and for every element $b \in B$ there exists q such that $b^q \in f(A)$. Then $\mathrm{Spec}\,(f) : \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ is a homeomorphism (recall that this map sends the prime ideal $Q \in \mathrm{Spec}\,(B)$ to the contraction $f^{-1}(Q)$ of Q to A). The inverse maps $P \in \mathrm{Spec}\,(A)$ to the radical of PB, which is the unique prime ideal of B lying over P.*

Proof. Since the induced map $\mathrm{Spec}\,(A) \to \mathrm{Spec}\,(A/J)$ is a homeomorphism whenever J is an ideal whose elements are nilpotent, and the unique prime of A/J lying over $P \in \mathrm{Spec}\,(A)$ is P/J, the image of P in A/J, there is no loss of generality in considering instead the induced map $A_{\mathrm{red}} \to B_{\mathrm{red}}$, which is injective. We therefore assume that A and B are reduced, and, by replacing A by its image, we may also assume that $A \subseteq B$. Then $A \hookrightarrow B$ is an integral extension, since every $b \in B$ has a power in A, and it follows that there is a prime ideal Q of S lying over a given prime P of A. If $u \in Q$, then $u^q \in A$ for some q, and so $u^q \in Q \cap A = P$. It follows that $Q \subseteq \mathrm{Rad}\,(PB)$, and since Q is a radical ideal containing PB, we have that $Q = \mathrm{Rad}\,(PB)$. Therefore, as claimed in the statement of the Proposition, we have that $\mathrm{Rad}\,(PB)$ is the unique prime ideal of S lying over P. This shows that $\mathrm{Spec}\,(f)$ is bijective. To show that $g = \mathrm{Spec}\,(f)$ is a homeomorphism, it suffices to show that its inverse is continuous, i.e., that g maps closed sets to closed sets. But for any $b \in B$, we may choose q so that $b^q \in A$, and then

$$g\big(\mathcal{V}(bB)\big) = \mathcal{V}(b^q A) \subseteq \mathrm{Spec}\,(A). \qquad \square$$

The Radu-André Theorem is valid even when R is not reduced. In this case, we do not want to use the notation $R^{1/q}$. Instead, we let $R^{(e)}$ denote R viewed as an R algebra via the structural homomorphism $F^e : R \to R$. We restate the result using this notation.

Theorem (Radu-André). *Let R and S be F-finite rings of prime characteristic $p > 0$ such that $R \to S$ is flat with geometrically regular fibers. Then for all e, $R^{(e)} \otimes_R S \to S^{(e)}$ is faithfully flat.*

Proof. Let $T_e = R^{(e)} \otimes_R S$. Consider the maps $S \to T_e \to S^{(e)}$. Any element in the kernel of $S \to S^{(e)}$ is nilpotent. It follows that this is also true of any element in the kernel of $S \to T_e$. Note that every element of T_e has q th power in the image of S, since $(r \otimes s)^q = r^q \otimes s^q = 1 \otimes r^q s^q$. It follows that $\mathrm{Spec}\,(S^{(e)}) \to \mathrm{Spec}\,(T_e) \to \mathrm{Spec}\,(S)$ are homeomorphisms. Hence, if $S^{(e)}$ is flat over T_e, then it is faithfully flat over T_e.

It is easy to see that geometric regularity is preserved by localization of either ring, and the issue of flatness is local on the primes of $S^{(e)}$ and their contractions to T_e. Localizing $S^{(e)}$ at a prime gives the same result as localizing at the contraction of that prime to S. It follows that we may replace S by a typical localization S_Q and R by R_P where P is the contraction of S ot R. Thus, we may assume that $(R,\, m,\, K)$ is local, and that $R \to S$ is a local homomorphism of local rings. Evidently, $S^{(e)}$ and $R^{(e)}$ are local as well, and it follows from the remarks in the first paragraph that the maps $S \to T_e \to S^{(e)}$ are also local.

Let $m^{(e)}$ be the maximal ideal of $R^{(e)}$: of course, if we identify $R^{(e)}$ with the ring R, then $m^{(e)}$ is identified with the maximal ideal m of R.

We shall now prove that $A = T_e \to S^{(e)} = B$ is flat using the local criterion for flatness, taking $I = m^{(e)}T_e$. Note that since $R \to S$ is flat, so is $R^{(e)} \to R^{(e)} \otimes_R S = T_e$. Therefore, $m^{(e)}T_e \cong m^{(e)} \otimes_R S$. The expansion of I to $B = S^{(e)}$ may be identified with $m^{(e)}S^{(e)}$, and since $R^{(e)} \to S^{(e)}$ as a map of rings is the same as $R \to S$, we have that $S^{(e)}$ is flat over $R^{(e)}$, and we may identify $m^{(e)}S^{(e)}$ with $m^{(e)} \otimes_{R^{(e)}} S^{(e)}$.

There are two things to check. One is that B/I is flat over A/I, which says that $S^{(e)}/(m^{(e)} \otimes_{R^{(e)}} S^{(e)})$ is flat over $(R^{(e)} \otimes_R S)/(m^{(e)} \otimes_R S)$. The former may be identified with $(S/mS)^{(e)}$, and the latter with $K^{(e)} \otimes_K (S/mS)$, since $R^{(e)}/m^{(e)}$ may be identified with $K^{(e)}$. Since R is F-finite, so is K, and it follows that $K^{(e)} \cong K^{1/q}$ is a finite purely inseparable extension of K. Since the fiber $K \to K \otimes_R S = S/mS$ is geometrically regular, we have that $K^{(e)} \otimes_R (S/mS) \cong K^{(e)} \otimes_K (S/mS)$ is regular and, in particular, reduced. Since it is purely inseparable over the regular local ring S/mS we have from the Proposition on p. 2 that

$$K^{(e)} \otimes_K (S/mS) \cong (S/mS)[K^{1/q}].$$

is a local ring. Hence, it is a regular local ring.

We have as well that $(S/mS)^{(e)} \cong (S/mS)^{1/q}$ is regular, since S/mS is, and is a module-finite extension of $(S/mS)[K^{1/q}]$. Thus, $B/IB = (S/mS)^{1/q}$ is module-finite local and Cohen-Macaulay over $A/IA = (S/mS)[K^{1/q}]$, which is regular local. By the Lemma on p. 8 of the Lecture Notes of September 8, B/IB is free over A/I, and therefore flat.

Finally, we need to check that $I \otimes_A B \twoheadrightarrow IB$ is an isomorphism, and this the map

$$\phi : (m^{(e)} \otimes_R S) \otimes_{R^{(e)} \otimes_R S} S^{(e)} \twoheadrightarrow m^{(e)} \otimes_{R^{(e)}} S^{(e)}.$$

The map takes $(u \otimes s) \otimes v$ to $u \otimes (sv)$. We prove that ϕ is injective by showing that it has an inverse. There is an $R^{(e)}$-bilinear map

$$m^{(e)} \times S^{(e)} \to (m^{(e)} \otimes_R S) \otimes_{R^{(e)} \otimes_R S} S^{(e)}$$

that sends $(u,\, v) \mapsto (u \otimes 1) \otimes v$. This induces a map

$$\psi : m^{(e)} \otimes_{R^{(e)}} S^{(e)} \to (m^{(e)} \otimes_R S) \otimes_{R^{(e)} \otimes_R S} S^{(e)}$$

and it is straightforward to see that $\psi \circ \phi$ sends

$$(u \otimes s) \otimes v \mapsto (u \otimes 1) \otimes (sv) = (u \otimes s) \otimes v,$$

and that $\phi \circ \psi$ sends $u \otimes v$ to itself. $\square$

Note that if a Noetherian ring R is reduced and $R \to S$ is flat with reduced fibers over the minimal primes of R, then S is reduced. (Because nonzerodivisors in R are nonzerodivisors on S, we can replace R by its total quotient ring, which is a product of fields, and S becomes the product of the fibers over the minimal primes of R.) Hence, if $R \to S$ is flat with geometrically regular (or even reduced) fibers and R is reduced, so is S. This is used several times in the sequel.

Theorem. *If $R \to S$ is a flat map of F-finite rings of prime characteristic $p > 0$ with geometrically regular fibers and R is strongly F-regular then so is S.*

Proof. We can choose $c \in R^\circ$ such that R_c is regular, and then we know that there is an R-linear map $\theta : R^{1/q} \to R$ sending $c^{1/q} \mapsto 1$. Now $R_c \to S_c$ is flat with regular fibers and R_c is regular, so that S_c is regular as well. By Theorem at the bottom of p. 6 of the Lecture Notes of September 24, it suffices to show that there is an S-linear map $S^{1/q} \to S$ such that $c^{1/q} \mapsto 1$. Let $\theta' = \theta \otimes_R \mathbf{1}_S : R^{1/q} \otimes_R S \to S$, so that θ' is an S-linear map such that $\theta'(c^{1/q} \otimes 1) = 1$. By the Corollary on p. 3 of the Lecture Notes of September 28, the inclusion $R^{1/q} \otimes_R S \to S^{1/q}$, which takes $c^{1/q} \otimes 1$ to $c^{1/q}$, has a splitting $\alpha : S^{1/q} \to R^{1/q} \otimes_R S$ that is linear over $R^{1/q} \otimes_R S$. Hence, α is also S-linear, and $\theta' \circ \alpha$ is the required S-linear map from $S^{1/q}$ to S. $\square$

We also can improve our result on the existence of big test elements now.

Theorem. *Let R be a reduced F-finite ring of prime characteristic $p > 0$ and let $c \in R^\circ$ be such that R_c is strongly F-regular. Also assume that there is an R-linear map $R^{1/p} \to R$ that sends 1 to c. If S is F-finite and flat over R with geometrically regular fibers, then the image of c^3 in S is a big test element for S.*

In particular, for every element c as above, c^3 is a completely stable big test element.

Hence, every element c of R° such that R_c is strongly F-regular has a power that is a completely stable big test element, and remains a completely stable big test element after every geometrically regular base change to an F-finite ring.

Proof. By the Theorem at the bottom of p. 1 of the Lecture Notes of September 26, to prove the result asserted in the first paragraph it suffices to show that the image of c in S has the same properties: because the map is flat, the image is in S°, and so it suffices to show that S_c is strongly F-regular and that there is an S-linear map $S^{1/p} \to S$ such that the value on 1 is the image of c in S. But the map $R_c \to S_c$ is flat, R_c is strongly F-regular, and the fibers are a subset of the fibers of the map $R \to S$ corresponding to primes of R not containing c. Hence, the fibers are geometrically regular, and so we can conclude that S_c is strongly F-regular. We have an R-linear map $R^{1/p} \to R$ that sends $1 \mapsto c$. We may apply $\_ \otimes_R \mathbf{1}_S$ to get a map $R^{1/p} \otimes_R S \to S$ sending 1 to the image of c, and then compose with a splitting of the inclusion $R^{1/p} \otimes_R S \to S^{1/p}$ to get the required map.

The statement of the second paragraph now follows because a localization map is geometrically regular, and F-finite rings are excellent, so that the map from a local ring to its completion is geometrically regular as well.

To prove the third statement note that whenever R_c is strongly F-regular, there is a map $R^{1/p} \to R$ whose value on 1 is a power of c: this is a consequence of the first Lemma on p. 1 of the Lecture Notes of September 26. $\square$

Mapping cones

Let $B_\bullet$ and $A_\bullet$ be complexes of R-modules with differentials $\delta_\bullet$ and $d_\bullet$, respectively. We assume that they are indexed by $\mathbb{Z}$, although in the current application that we have in mind they will be left complexes, i.e., all of the negative terms will be zero. Let $\phi_\bullet$ be a map of complexes, so that for every n we have $\phi_n : B_n \to A_n$, and all the squares

$$\begin{array}{ccc} A_n & \xrightarrow{d_n} & A_{n-1} \\ {\scriptstyle \phi_n}\uparrow & & \uparrow{\scriptstyle \phi_{n-1}} \\ B_n & \xrightarrow{\delta_n} & B_{n-1} \end{array}$$

commute. The *mapping cone* $\mathcal{C}_\bullet^{\phi_\bullet}$ of $\phi_\bullet$ is defined so that $\mathcal{C}_n^{\phi_\bullet} := A_n \oplus B_{n-1}$ with the differential that is simply d_n on A_n and is $(-1)^{n-1}\phi_{n-1} \oplus \delta_{n-1}$ on B_{n-1}. Thus, under the differential in the mapping cone,

$$a_n \oplus b_{n-1} \mapsto \big(d_n(a_n) + (-1)^{n-1}\phi(b_{n-1})\big) \oplus \delta_{n-1}(b_{n-1}).$$

If we apply the differential a second time, we obtain

$$\Big(d_{n-1}\big(d_n(a_n) + (-1)^{n-1}\phi_{n-1}(b_{n-1})\big) + (-1)^{n-2}\phi_{n-2}\delta_{n-1}(b_{n-1})\Big) \oplus \delta_{n-2}\delta_{n-1}(b_{n-1}),$$

which is 0, and so we really do get a complex. We frequently omit the superscript ${}^{\phi_\bullet}$, and simply write $\mathcal{C}_\bullet$ for $\mathcal{C}_\bullet^{\phi_\bullet}$.

Note that $A_\bullet \subseteq \mathcal{C}_\bullet$ is a subcomplex. The quotient complex is isomorphic with $B_\bullet$, except that degrees are shifted so that the degree n term in the quotient is B_{n-1}. This leads to a long exact sequence of homology:

$$\cdots \to H_n(A_\bullet) \to H_n(\mathcal{C}_\bullet) \to H_{n-1}(B_\bullet) \to H_{n-1}(A_\bullet) \to \cdots .$$

One immediate consequence of this long exact sequence is the following fact.

Proposition. *Let $\phi_\bullet : B_\bullet \to A_\bullet$ be a map of left complexes. Suppose that $A_\bullet$ and $B_\bullet$ are acyclic, and that the induced map of augmentations $H_0(B_\bullet) \to H_0(A_\bullet)$ (which may also be described as the induced map $B_0/\delta_0(B_1) \to A_0/d_0(A_1)$) is injective. Then the mapping cone is an acyclic left complex, and its augmentation is $A_0/\big(d(A_1) + \phi_0(B_0)\big)$.* $\square$

The Koszul complex

The Koszul complex $\mathcal{K}_\bullet(x_1, \ldots, x_n;\, R)$ of a sequence of elements $x_1, \ldots, x_n \in R$ on R may be defined as an iterated mapping cone as follows. Let $\mathcal{K}_\bullet(x_1;\, R)$ denote the left complex in which $\mathcal{K}_1(x_1;\, R) = Ru_1$, a free R-module, $\mathcal{K}_0(x_1,\, R) = R$, and the map is such that $u_1 \mapsto x_1$. I.e., we have $0 \longrightarrow Ru_1 \xrightarrow{u_1 \mapsto x_1} R \to 0$.

Then we may define $\mathcal{K}_\bullet(x_1, \ldots, x_n;\, R)$ recursively as follows. If $n > 1$, multiplication by x_n (in every degree) gives a map of complexes

$$\mathcal{K}_\bullet(x_1, \ldots, x_{n-1};\, R) \xrightarrow{x_n \cdot} \mathcal{K}_\bullet(x_1, \ldots, x_{n-1}),$$

and we let $\mathcal{K}_\bullet(x_1, \ldots, x_n;\, R)$ be the mapping cone of this map.

We may prove by induction that $\mathcal{K}_n(x_1, \ldots, x_n;\, R)$ is a free complex of length n in which the degree j term is isomorphic with the free R-module on $\binom{n}{j}$ generators, $0 \le j \le n$. Even more specifically, we show that we may identify $\mathcal{K}_j(x_1, \ldots, x_n;\, R)$ with the free module on generators u_σ indexed by the j element subsets σ of $\{1, 2, \ldots, n\}$ in such a way that if $\sigma = \{i_1, \ldots, i_j\}$ with $1 \le i_1 < \cdots < i_j \le n$, then

$$du_\sigma = \sum_{t=1}^{j} (-1)^{t-1} x_{i_t} u_{\sigma - \{i_t\}}.$$

We shall use the alternative notation $u_{i_1 i_2 \cdots i_j}$ for u_σ in this situation. We also identify $u_\emptyset$, the generator of $\mathcal{K}_0(x_1, \ldots, x_n;\, R)$, with $1 \in R$.

To carry out the inductive step, we assume that $A_\bullet = \mathcal{K}_\bullet(x_1, \ldots, x_{n-1};\, R)$ has the specified form. We think of this complex as the target of the map multiplication by x_n, and index its generators by the subsets of $\{1, 2, \ldots, n-1\}$. This complex will be a subcomplex of $\mathcal{K}_\bullet(x_1, \ldots, x_n;\, R)$. We index the generators of the complex $B_\bullet$, which will be the domain for the map given by multiplication by x_n, and which is also isomorphic to

$\mathcal{K}_\bullet(x_1, \dots, x_{n-1}; R)$, by using the free generator $u_{\sigma\cup\{n\}}$ to correspond to u_σ. In this way, it is clear that $\mathcal{K}_\bullet(x_1, \dots, x_n;\, R)$ is free, and we have indexed its generators in degree j precisely by the j element subsets of $\{1,\, 2,\, \dots,\, n\}$. It is straightforward to check that the differential is as described above.

We can define the *Koszul complex* of $x_1, \dots, x_n \in R$ on an R-module M, which we denote $\mathcal{K}_\bullet(x_1, \dots, x_n;\, M)$, in two ways. One is simply as $\mathcal{K}_\bullet(x_1, \dots, x_n;\, R) \otimes_R M$. The second is to let $\mathcal{K}_\bullet(x_1; M)$ be the complex $0 \to M \otimes_R Ru_1 \to M \to 0$, and then to let $\mathcal{K}_\bullet(x_1, \dots, x_n;\, M)$ be the mapping cone of multiplication by x_n mapping the complex $\mathcal{K}_\bullet(x_1, \dots, x_{n-1};\, M)$ to itself, just as we did in the case $M = R$. It is quite easy to verify that these two constructions give isomorphic results: in fact, quite generally, $\_ \otimes_R M$ commutes with the mapping cone construction on maps of complexes of R-modules.