Math 711: Lecture of October 5, 2007

More on mapping cones and Koszul complexes

Let $\phi_\bullet : B_\bullet \to A_\bullet$ be a map of complexes that is injective. We shall write $d_\bullet$ for the differential on $A_\bullet$ and $\delta_\bullet$ for the differential on $B_\bullet$. Then we may form a quotient complex $Q_\bullet$ such that $Q_n = B_n/\phi_n(A_n)$ for all n, and the differential on $Q_\bullet$ is induced by the differential on $B_\bullet$. Let $\mathcal{C}_\bullet$ be the mapping cone of $\phi_\bullet$.

Proposition. *With notation as in the preceding paragraph, $H_n(\mathcal{C}_\bullet) \cong H_n(Q_\bullet)$ for all n.*

Proof. We may assume that every ϕ_n is an inclusion map. A cycle in Q_n is represented by an element $z \in A_n$ whose boundary $d_n z$ is 0 in $A_{n-1}/\phi_{n-1}(B_{n-1})$. This means that $d_n z = \phi_{n-1}(b)$ for some $b \in B_{n-1}$. (Once we have specified z there is at most one choice of b, by the injectivity of ϕ_{n-1}.) The boundaries in Q_n are represented by the elements $d_{n+1}(A_{n+1}) + \phi_n(B)$. Thus,

$$H_n(Q_\bullet) \cong \frac{d_n^{-1}\bigl(\phi_{n-1}(B_{n-1})\bigr)}{d_{n+1}(A_{n+1}) + \phi_n(B_n)}.$$

A cycle in $\mathcal{C}_n$ is represented by a sum $z \oplus b'$ such that

$$\bigl(d_n(z) + (-1)^{n-1}\phi_{n-1}(b')\bigr) \oplus \delta_{n-1}(b') = 0$$

Again, this element is uniquely determined by z, which must satisfy $d_n(z) \in \phi_{n-1}(B_{n-1})$. b' is then uniquely determined as $(-1)^n b$ where $b \in B_{n-1}$ is such that $\phi_{n-1}(b) = d_n(z)$. Such an element b is automatically killed by δ_{n-1}, since

$$\phi_{n-2}\delta_{n-1}(b) = d_{n-1}\phi_{n-1}(b) = d_{n-1}d_n(z) = 0,$$

and ϕ_{n-2} is injective. A boundary in $\mathcal{C}_n$ has the form

$$\bigl(d_{n+1}(a) + (-1)^n\phi_n(b_n)\bigr) \oplus \delta_n b_n.$$

This shows that

$$H_n(\mathcal{C}_\bullet) \cong \frac{d_n^{-1}\bigl(\phi_{n-1}(B_{n-1})\bigr)}{d_{n+1}(A_{n+1}) + \phi_n(B_n)},$$

as required. $\square$

Corollary. *Let $\underline{x} = x_1, \ldots, x_n \in R$ be elements such that x_n is not a zerodivisor on the R-module M. Let $\underline{x}^- = x_1, \ldots, x_{n-1}$, i..e., the result of omitting x_n from the sequence. Then $H_i(\underline{x};\, M) \cong H_i(\underline{x}^-;\, M/x_nM)$ for all i.*

Proof. We apply that preceding Proposition with $A_\bullet = B_\bullet = \mathcal{K}_\bullet(\underline{x}^-;\, M)$, and ϕ_i given by multiplication by x_n in every degree i. Since every term of $\mathcal{K}_\bullet(\underline{x}^-;\, M)$ is a finite direct sum of copies of M, the maps ϕ_i are injective. The mapping cone, which is $\mathcal{K}_\bullet(\underline{x};\, M)$, therefore has the same homology as the quotient complex, which may be identified with

$$\mathcal{K}_\bullet(\underline{x}^-,\, M) \otimes (R/x_nR) \cong \mathcal{K}_\bullet(\underline{x}^-;\, R) \otimes_R M \otimes_R R/x_nR \cong \mathcal{K}_\bullet(\underline{x}^-;\, R) \otimes_R (M/x_nM)$$

which is $\mathcal{K}_\bullet(\underline{x}^-;\, M/x_nM)$, and the result follows. $\square$

We also observe:

Proposition. *Let $\phi_\bullet : B_\bullet \to A_\bullet$ be any map of complexes and let $\mathcal{C}_\bullet$ be the mapping cone. In the long exact sequence*

$$\cdots \to H_n(A_\bullet) \to H_n(\mathcal{C}_\bullet) \to H_{n-1}(B_\bullet) \xrightarrow{\partial_{n-1}} H_{n-1}(A_\bullet) \to \cdots$$

the connecting homomorphism ∂_{n-1} is induced by $(-1)^{n-1}\phi_{n-1}$.

Proof. We follow the prescription for constructing the connecting homomorphism. Let $b \in B_{n-1}$ be a cycle in B_{n-1}. We lift this cycle to an element of $\mathcal{C}_n$ that maps to it: one such lifting is $0 \oplus b$ (the choice of lifting does not affect the result). We now apply the differential in the mapping cone $\mathcal{C}_\bullet$ to the lifting: this gives

$$(-1)^{n-1}\phi_{n-1}(b) \oplus \delta_{n-1}(b) = (-1)^{n-1}\phi_{n-1}(b) \oplus 0,$$

since b was a cycle in B_{n-1}. Call the element on the right α. Finally, we choose an element of A_{n-1} that maps to α: this gives $(-1)^{n-1}\phi_{n-1}(b)$, which represents the value of $\partial_{n-1}([b])$, as required. $\square$

Corollary. *Let $\underline{x} = x_1, \ldots, x_n \in R$ be arbitrary elements. Let $\underline{x}^- = x_1, \ldots, x_{n-1}$, i..e., the result of omitting x_n from the sequence. Let M be any R-module. Then there are short exact sequences*

$$0 \to \frac{H_i(\underline{x}^-;\, M)}{x_nH_i(\underline{x}^-;\, M)} \to H_i(\underline{x};\, M) \to \mathrm{Ann}_{H_{i-1}(\underline{x}^-;\, M)}x_n \to 0$$

for every integer i.

Proof. By the preceding Proposition, the long exact sequence for the homology of the mapping cone of the map of complexes

$$\mathcal{K}_\bullet(\underline{x}^-;\, M) \xrightarrow{x_n\cdot} \mathcal{K}_\bullet(\underline{x}^-;\, M)$$

has the form

$$\cdots \longrightarrow H_i(\underline{x}^-;\, M) \xrightarrow{(-1)^i x_n \cdot} H_i(\underline{x}^-;\, M) \longrightarrow H_i(\underline{x};\, M)$$

$$\longrightarrow H_{i-1}(\underline{x}^-;\, M) \xrightarrow{(-1)^{i-1} x_n \cdot} H_{i-1}(\underline{x}^-;\, M) \longrightarrow \cdots$$

Since the maps given by multiplication by x_n and by $-x_n$ have the same kernel and cokernel, this sequence implies the existence of the short exact sequences specified in the statement of the Theorem. $\square$

The cohomological Koszul complex

Notice that if P is a finitely generated projective module over a ring R, $\_^*$ denotes the functor that sends $N \mapsto \mathrm{Hom}_R(N,\, R)$, and M is any module, then there is a natural isomorphism

$$\mathrm{Hom}_R(P,\, M) \cong P^* \otimes_R M$$

such that the inverse map η_P is defined as follows: η_P is the linear map induced by the R-bilinear map B_P given by $B_P(g,\, u)(v) = g(v)u$ for $g \in P^*$, $u \in M$, and $v \in P$. It is easy to check that

(1) $\eta_{P\oplus Q} = \eta_P \oplus \eta_Q$ and

(2) that η_R is an isomorphism.

It follows at once that

(3) η_{R^n} is an isomorphism for all $n \in \mathbb{N}$.

For any finitely generated projective module P we can choose Q such that $P \oplus Q \cong R^n$, and then, since $\eta_P \oplus \eta_Q$ is an isomorphism, it follows that

(4) η_P is an isomorphism for every finitely generated projective module P.

If R is a ring, M an R-module, and $\underline{x} = x_1,\, \ldots\,, x_n \in R$, the *cohomological Koszul complex* $\mathcal{K}^\bullet(\underline{x};\, M)$, is defined as

$$\mathrm{Hom}_R\big(\mathcal{K}_\bullet(\underline{x};\, R),\, M\big),$$

and its cohomology, called *Koszul cohomology*, is denoted $H^\bullet(\underline{x};\, M)$. The cohomological Koszul complex of R (and, it easily follows, of M) is isomorphic with the homological Koszul complex numbered "backward," but this is not quite obvious: one needs to make sign changes on the obvious choices of bases to get the isomorphism.

To see this, take the elements $u_{j_1,\dots,j_i}$ with $1 \le j_1 < \cdots < j_i \le n$ as a basis for $\mathcal{K}_i = \mathcal{K}_i(\underline{x};\, R)$. We continue to use the notation $\_^*$ to indicate the functor $\mathrm{Hom}_R(\_,\, R)$. We want to set up isomorphisms $\mathcal{K}^*_{n-i} \cong \mathcal{K}_i$ that commute with the differentials.

Note that there is a bijection between the two free bases for $\mathcal{K}_i$ and $\mathcal{K}_{n-i}$ as follows: given $1 \le j_1 < \cdots < j_i \le n$, let $k_1, \dots, k_{n-i}$ be the elements of the set

$$\{1,\, 2,\, \dots,\, n\} - \{j_1,\, \dots,\, j_i\}$$

arranged in increasing order, and let $u_{j_1,\dots,j_i}$ correspond to $u_{k_1,\dots,k_{n-i}}$ which we shall also denote as $v_{j_1,\dots,j_i}$.

When a free R-module G has free basis $b_1, \dots, b_t$, this determines what is called a *dual basis* $b'_1, \dots, b'_t$ for G^*, where b'_j is the map $G \to R$ that sends b_j to 1 and kills the other elements in the free basis. Thus, $\mathcal{K}^*_{n-i}$ has basis $v'_{j_1,\dots,j_i}$. However, when we compute the value of the differential d^*_{n-i+1} on $v'_{j_1,\dots,j_i}$, while the coefficient of $v'_{h_1,\dots,h_{i-1}}$ does turn out to be zero unless the elements $h_1 < \cdots < h_{i-1}$ are included among the j_i, if the omitted element is j_t then the coefficient of $v'_{h_1,\dots,h_{i-1}}$ is

$$d^*_{n-i+1}(v'_{j_1,\dots,j_i})(v_{h_1,\dots,h_{i-1}}) = v'_{j_1,\dots,j_i}\big(d_{n-i+1}(v_{h_1,\dots,h_{i-1}})\big),$$

which is the coefficient of $v_{j_1,\dots,j_i}$ in $d_{n-i+1}(v_{h_1,\dots,h_{i-1}})$.

Note that the complement of $\{j_1, \dots, j_i\}$ in $\{1,\, 2,\, \dots,\, n\}$ is the same as the complement of $\{h_1, \dots, h_{i-1}\}$ in $\{1,\, 2,\, \dots,\, n\}$, except that one additional element, j_t, is included in the latter. Thus, the coefficient needed is $(-1)^{s-1}x_{j_t}$, where $s-1$ is the number of elements in the complement of $\{h_1, \dots, h_{i-1}\}$ that precede j_t. The signs don't match what we get from the differential in $\mathcal{K}_\bullet(\underline{x};\, R)$: we need a factor of $(-1)^{(s-1)-(t-1)}$ to correct (note that $t-1$ is the number of elements in $j_1, \dots, j_i$ that precede j_t). This sign correction may be written as $(-1)^{(s-1)+(t-1)}$, and the exponent is $j_t - 1$, the total number of elements preceding j_t in $\{1, 2, \dots, n\}$. This sign implies that the signs will match the ones in the homological Koszul complex if we replace every v'_{j_i} by $(-1)^\Sigma v'_{j_i}$, where $\Sigma = \sum_{t=1}^{i}(j_t - 1)$. This completes the proof. $\square$

This duality enables us to compute Ext using Koszul homology, and, hence, Tor in certain instances:

Theorem. *Let $\underline{x} = x_1, \dots, x_n$ be a possibly improper regular sequence in a ring R and let M be any R-module. Then*

$$\mathrm{Ext}^i_R(R/(\underline{x})R;\, M) \cong H^i(\underline{x};\, M) \cong H_{n-i}(\underline{x};\, M) \cong \mathrm{Tor}^R_{n-i}(R/(\underline{x})R,\, M).$$

Proof. Because the Koszul complex on the x_i is a free resolution of $R/(\underline{x})R$, we may use it to calculate $\mathrm{Ext}^j(R/(\underline{x})R,\, M)$: this yields the leftmost isomorphism. The middle isomorphism now follows from the self-duality of the Koszul complex proved above, and we have already proved that the Koszul homology yields Tor when $\underline{x}$ is a regular sequence in R: this is simply because we may use again that $\mathcal{K}_\bullet(\underline{x};\, R)$ is a free resolution of $R/(\underline{x})R$. $\square$

Depth and Ext

When $R \to S$ is a homomorphism of Noetherian rings, N is a finitely generated R-module, and M is a finitely generated S-module, the modules $\mathrm{Ext}^j_R(N,\, M)$ are finitely generated S-modules. One can see this by taking a left resolution $G_\bullet$ of N by finitely generated free R-modules, so that

$$\mathrm{Ext}^j_R(N,\, M) = H^j\big(\mathrm{Hom}_R(G_\bullet,\, M)\big).$$

Since each term of $\mathrm{Hom}_R(G_\bullet,\, M)$ is a finite direct sum of copies of M, the statement follows.

If I is an ideal of R such that $IM \neq M$, then any regular sequence in I on M can be extended to a maximal such sequence that is necessarily finite. To see that we cannot have an infinite sequence $x_1,\, \ldots\, , x_n,\, \ldots \in I$ that is a regular sequence on M we may reason as follows. Because R is Noetherian, the ideals $J_n = (x_1,\, \ldots\, , x_n)R$ must be eventually constant. Alternatively, we may argue that because M is Noetherian over S, the submodules J_nM must be eventually constant. In either case, once $J_nM = J_{n+1}M$ we have that $x_{n+1}M \subseteq J_nM$, and so the action of x_{n+1} on M/J_nM is 0. Since $J_n \subseteq I$ and $IM \neq M$, we have that $M/J_nM \neq 0$, and this is a contradiction, since x_{n+1} is supposed to be a nonzerodivisor on M/J_nM. We shall show that maximal regular sequences on M in I all have the same length, which we will then define to be the *depth* of M on I.

The following result will be the basis for our treatment of depth.

Theorem. *Let $R \to S$ be a homomorphism of Noetherian rings, let $I \subseteq R$ be an ideal and let N be a finitely generated R-module with annihilator I. Let M be a finitely generated S-module with annihilator $J \subseteq S$.*

(a) *The support of $N \otimes_R M$ is $\mathcal{V}(IS + J)$. Hence, $N \otimes_R M = 0$ if and only if $IS + J = S$. In particular, $M = IM$ if and only if $IS + J = S$.*

(b) *If $IM \neq M$, then there are finite maximal regular sequences $x_1,\, \ldots\, , x_d$ on M in I. For any such maximal regular sequence, $\mathrm{Ext}^i_R(N,\, M) = 0$ if $i < d$ and $\mathrm{Ext}^d_R(N,\, M) \neq 0$. In particular, these statements hold when $N = R/I$. Hence, any two maximal regular sequences in I on M have the same length.*

(c) *$IM = M$ if and only if $\mathrm{Ext}^i_R(N,\, M) = 0$ for all i. In particular, this statement holds when $N = R/I$.*

Proof. (a) $N \otimes_R M$ is clealy killed by J and by I. Since it is an S-module, it is also killed by IS and so it is killed by $IS + J$. It follows that any prime in the support must contain

$IS+J$. Now suppose that $Q \in \mathrm{Spec}\,(S)$ is in $\mathcal{V}(IS+J)$, and let P be the contraction of Q to R. It suffices to show that $(N \otimes_R M)_Q \neq 0$, and so it suffices to show that $N_P \otimes_{R_P} M_Q \neq 0$. Since $I \subseteq P$, $N_P \neq 0$ and N_P/PN_P is a nonzero vector space over $\kappa = R_P/PR_P$: call it κ^s, where $s \geq 1$. M_Q maps onto $M_Q/QM_Q = \lambda^t$, where $\lambda = S_Q/QS_Q$, is a field, $t \geq 1$, and we have $\kappa \hookrightarrow \lambda$. But then we have

$$(N \otimes_R M)_Q \cong N_P \otimes_{R_P} M_Q \twoheadrightarrow \kappa^s \otimes_{R_P} \lambda^t \cong \kappa^s \otimes_\kappa \lambda^t \cong (\kappa \otimes_\kappa \lambda)^{st} \cong \lambda^{st} \neq 0,$$

as required. The second statement in part (a) is now clear, and the third is the special case where $N = R/I$.

Now assume that $M \neq IM$, and choose any maximal regular sequence $x_1, \ldots, x_d \in I$ on M. We shall prove by induction on d that $\mathrm{Ext}^i_R(N, M) = 0$ for $i < d$ and that $\mathrm{Ext}^d_R(N, M) \neq 0$.

First suppose that $d = 0$. Let $Q_1, \ldots, Q_h$ be the associated primes of M in S. Let P_j be the contraction of Q_j to R for $1 \leq j \leq h$. The fact that $\mathrm{depth}_I M = 0$ means that I consists entirely of zerodivisors on M, and so I maps into the union of the Q_j. This means that I is contained in the union of the P_j, and so I is contained in one of the P_j: called it $P_{j_0} = P$. Choose $u \in M$ whose annihilator in S is Q_{j_0}, and whose annihilator in R is therefore P. It will suffice to show that $\mathrm{Hom}_R(N, M) \neq 0$, and therefore to show that its localization at P is not 0, i.e., that $\mathrm{Hom}_{R_P}(N_P, M_P) \neq 0$. Since P contains $I = \mathrm{Ann}_R N$, we have that $N_P \neq 0$. Therefore, by Nakayama's lemma, we can conclude that $N_P/PN_P \neq 0$. This module is then a nonzero finite dimensional vector space over $\kappa_P = R_P/PR_P$, and we have a surjection $N_P/PN_P \twoheadrightarrow \kappa_P$ and therefore a composite surjection $N_P \twoheadrightarrow \kappa_P$. Consider the image of $u \in M$ in M_P. Since $\mathrm{Ann}_R u = P$, the image v of $u \in M_P$ is nonzero, and it is killed by P. Thus, $\mathrm{Ann}_{R_P} v = PR_P$, and it follows that v generates a copy of κ_P in M_P, i.e., we have an injection $\kappa_P \hookrightarrow M_P$. The composite map $N_P \twoheadrightarrow \kappa_P \hookrightarrow M_P$ gives a nonzero map $N_P \to M_P$, as required.

Finally, suppose that $d > 0$. Let $x = x_1$, which is a nonzerodivisor on M. Note that $x_2, \ldots, x_d \in I$ is a maximal regular sequence on M/xM. Since $x \in I$, we have that x kills N. The short exact sequence $0 \to M \to M \to M/xM \to 0$ gives a long exact sequence for Ext when we apply $\mathrm{Hom}_R(N, \_)$. Because x kills N, it kills all of the Ext modules in this sequence, and thus the maps induced by multiplication by x are all 0. This implies that the long exact sequence breaks up into short exact sequences

$$(*_j) \quad 0 \to \mathrm{Ext}^j_R(N, M) \to \mathrm{Ext}^j_R(N, M/xM) \to \mathrm{Ext}^{j+1}_R(N, M) \to 0$$

We have from the induction hypothesis that the modules $\mathrm{Ext}^j_R(N, M/xM) = 0$ for $j < d-1$, and the exact sequence above shows that $\mathrm{Ext}^j_R(N, M) = 0$ for $j < d$. Moreover, $\mathrm{Ext}^{d-1}_R(N, M/xM) \neq 0$, and $(*_{d-1})$ shows that $\mathrm{Ext}^{d-1}_R(N, M/xM)$ is isomorphic with $\mathrm{Ext}^d_R(N, M)$.

The final statement in part (b) follows because the least exponent j for which, say, $\mathrm{Ext}^j_R(R/I, M) \neq 0$ is independent of the choice of maximal regular sequence.

It remains to prove part (c). If $IM \neq M$, we can choose a maximal regular sequence $x_1, \ldots, x_d$ on M in I, and then we know from part (b) that $\mathrm{Ext}^d_R(N, M) \neq 0$. On the other hand, if $IM = M$, we know that $IS + \mathrm{Ann}_R M = S$ from part (a), and this ideal kills every $\mathrm{Ext}^j_R(N, M)$, so that all of the Ext modules vanish. $\square$

If $R \to S$ is a map of Noetherian rings, M is a finitely generated S-module, and $IM \neq M$, we define $\mathrm{depth}_I M$, the *depth* of M on I, to be, equivalently, the length of *any* maximal regular sequence in I on M, or $\inf\{j \in \mathbb{Z} : \mathrm{Ext}^j_R(R/I, M) \neq 0\}$. If $IM = M$, we define the depth of M on I as $+\infty$, which is consistent with the Ext characterization.

Note the following:

Corollary. *With hypothesis as in the preceding Theorem,* $\mathrm{depth}_I M = \mathrm{depth}_{IS} M$. *Moreover, if R' is flat over R, e.g., a localization of R, then* $\mathrm{depth}_{IR'} R' \otimes_R M \geq \mathrm{depth}_I M$.

Proof. Choose a maximal regular sequence in I, say $x_1, \ldots, x_d$. These elements map to a regular sequence in IS. We may replace M by $M/(x_1, \ldots, x_d)M$. We therefore reduce to showing that when $\mathrm{depth}_I M = 0$, it is also true that $\mathrm{depth}_{IS} M = 0$. But it was shown in the proof of the Theorem above that that under the condition $\mathrm{depth}_I M = 0$ there is an element $u \in M$ whose annihilator is an associated prime $Q \in \mathrm{Spec}\,(S)$ of M that contains IS. The second statement follows from the fact that calculation of Ext_R commutes with flat base change when the first module is finitely generated over R. (One may also use the characterization in terms of regular sequences.) $\square$

We also note:

Proposition. *With hypothesis as in the preceding Theorem, let $\underline{x} = x_1, \ldots, x_n$ be generators of $I \subseteq R$. If $IM = M$, then all of the Koszul homology $H_i(\underline{x};\, M) = 0$. If $IM \neq M$, then $H_{n-i}(\underline{x};\, M) = 0$ if $i < d$, and $H_{n-d}(\underline{x};\, M) \neq 0$.*

Proof. We may map a Noetherian ring B containing elements $X_1, \ldots, X_n$ that form a regular sequence in B to R so that $X_i \mapsto x_i$, $1 \leq i \leq n$. For example, we may take $B = R[X_1, \ldots, X_n]$ and map to R using the R-algebra map that sends $X_i \mapsto x_i$, $1 \leq i \leq n$. Let $J = (X_1, \ldots, X_n)B$. Then $\mathrm{depth}_I M = \mathrm{depth}_J M$, and the latter is determined by the least integer j such that $\mathrm{Ext}^j_B(B/(\underline{X})B,\, M) \neq 0$. The result is now immediate from the Theorem at the bottom of p. 4. $\square$

Cohen-Macaulay rings and lifting while preserving height

Proposition. *A Noetherian ring R is Cohen-Macaulay if and only if for every proper ideal I of R,* $\mathrm{depth}_I R = \mathrm{height}\,(I)$.

Proof. Suppose that R is Cohen-Macaulay, and let I be any ideal of R. We use induction on $\mathrm{height}\,(I)$. If $\mathrm{height}\,(I) = 0$, then I is contained in a minimal prime of R, and so

$\mathrm{depth}_I R = 0$. Now suppose that height $(I) > 0$. Each prime in $\mathrm{Ass}\,(R)$ must be minimal: otherwise, we may localize at such a prime, which yields a Cohen-Macaulay ring of positive dimension such that every element of its maximal ideal is a zerodivisor, a contradiction. Since I is not contained in the union of the minimal primes, I is not contained in the union of the primes in $\mathrm{Ass}\,(R)$. Choose an element $x_1 \in I$ not in any minimal prime of R and, hence, not a zerodivisor on R. It follows that R/x_1R is Cohen-Macaulay, and the height of I drops exactly by one. The result now follows from the induction hypothesis applied to $I/x_1R \in R/x_1R$.

For the converse, we may apply the hypothesis with I a given maximal ideal m of height d. Then m contains a regular sequence of length d, say $x_1, \ldots, x_d$. This is preserved when we pass to R_m. The regular sequence remains regular in R_m, and so must be a system of parameters for R_m: killing a nonzerodivisor drops the dimension of a local ring by exactly 1. Hence, R_m is Cohen-Macaulay. $\square$

We also note:

Proposition. *Let R be a Noetherian ring and let $x_1, \ldots, x_d$ generate a proper ideal I of height d. Then there exist elements $y_1, \ldots, y_d \in R$ such that for every i, $1 \le i \le d$, $y_i \in x_i + (x_{i+1}, \ldots, x_d)R$, and for all i, $1 \le i \le d$, $y_1, \ldots, y_i$ generate an ideal of height i in R. Moreover, $(y_1, \ldots, y_d) = I$, and $y_d = x_d$.*

If R is Cohen-Macaulay, then $y_1, \ldots, y_d$ is a regular sequence.

Proof. We use induction on d. Note that by the coset form of the Lemma on prime avoidance, we cannot have that $x_1 + (x_2, \ldots, x_d)R$ is contained in the union of the minimal primes of R, or else $(x_1, \ldots, x_d)R$ has height 0. This enables us to pick $y_1 = x_1 + \Delta_1$ with $\Delta_1 \in (x_2, \ldots, x_d)R$ such that y_1 is not in any minimal prime of R. In case R is Cohen-Macaulay, this implies that y_1 is not a zerodivisor. It is clear that $(y_1, x_2, \ldots, x_d)R = I$. The result now follows from the induction hypothesis applied to the images of $x_2, \ldots, x_d$ in R/y_1R. $\square$

Note that even in the polynomial ring $K[x, y, z]$ the fact that three elements generate an ideal of height three does not imply that these elements form a regular sequence: $(1-x)y$, $(1-x)z$, x gives a counterexample.

Proposition. *Let R be a Noetherian ring, let $\mathfrak{p}$ be a minimal prime of R, and let $x_1, \ldots, x_d$ be elements of R such that $(x_1, \ldots, x_i)(R/\mathfrak{p})$ has height i, $1 \le i \le d$. Then there are elements $\delta_1, \ldots, \delta_d \in \mathfrak{p}$ such that if $y_i = x_i + \delta_i$, $1 \le i \le d$, then $(y_1, \ldots, y_i)R$ has height i, $1 \le i \le d$.*

Proof. We construct the δ_i recursively. Suppose that $\delta_1, \ldots, \delta_t$ have already been chosen: t may be 0. If $t < d$, we cannot have that $x_{t+1} + \mathfrak{p}$ is contained in the union of the minimal primes of $(y_1, \ldots, y_t)$. If that were the case, by the coset form of prime avoidance we would have that $x_{t+1}R + \mathfrak{p} \subseteq Q$ for one such minimal prime Q. Then Q has height at most t, but modulo $\mathfrak{p}$ all of $x_1, \ldots, x_{t+1}$ are in Q, so that height $(Q/\mathfrak{p}) \ge t + 1$, a contradiction. $\square$

The following result will be useful in proving the colon-capturing property for tight closure.

Lemma. *Let P be a prime ideal of height h in a Cohen-Macaulay ring S. Let $x_1, \ldots, x_{k+1}$ be elements of $R = S/P$ such that $(x_1, \ldots, x_k)R$ has height k in R while $(x_1, \ldots, x_{k+1})R$ has height $k+1$. Then we can choose elements $y_1, \ldots, y_h \in P$ and $z_1, \ldots, z_{k+1} \in S$ such that:*

(1) *$y_1, \ldots, y_h, z_1, \ldots, z_{k+1}$ is a regular sequence in S.*

(2) *The images of $z_1, \ldots, z_k$ in R generate the ideal $(x_1, \ldots, x_k)R$.*

(3) *The image of z_{k+1} in R is x_{k+1}.*

Proof. By the first Proposition on p. 8, we may assume without loss of generality that $x_1, \ldots, x_i$ generate an ideal of height i in R, $1 \leq i \leq k$. We also know this for $i = k+1$. Choose z_i arbitrarily such that z_i maps to x_i, $1 \leq i \leq k+1$. Choose a regular sequence $y_1, \ldots, y_h$ of length h in P. Then P is a minimal prime of $(y_1, \ldots, y_h)S$. By the second Proposition on p. 8 applied to the images of the of the z_i in $S/(y_1, \ldots, y_h)S$ with $\mathfrak{p} = P/(y_1, \ldots, y_h)S$, we may alter the z_i by adding elements of P so that the height of the image of the ideal generated by the images of $z_1, \ldots, z_i$ in $S/(y_1, \ldots, y_h)$ is i, $1 \leq i \leq k+1$. Since $S/(y_1, \ldots, y_h)S$ is again Cohen-Macaulay, it follows from the first Proposition on p. 8 that the images of the $z_1, \ldots, z_{k+1}$ modulo $(y_1, \ldots, y_h)S$ form a regular sequence. But this means that $y_1, \ldots, y_h, z_1, \ldots, z_{k+1}$ is a regular sequence. $\square$

Colon-capturing

We can now prove a result on the colon-capturing property of tight closure.

Theorem (colon-capturing). *Let R be a reduced Noetherian ring of prime characteristic $p > 0$ that is a homomorphic image of a Cohen-Macaulay ring. Let $x_1, \ldots, x_{k+1}$ be elements of R. Let I_t denote the ideal $(x_1, \ldots, x_t)R$, $0 \leq t \leq k+1$. Suppose that the image of the ideal I_k has height k modulo every minimal prime of R, and that the image of the ideal $I_{k+1}R$ has height $k+1$ modulo every minimal prime of R. Then:*

(a) $I_k :_R x_{k+1} \subseteq I_k^*$.

(b) *If R has a test element, $I_k^* :_R x_{k+1} \subseteq I_k^*$, i.e., x_{k+1} is not a zerodivisor on R/I_k^*.*

Proof. To prove part (a), note that it suffices to prove the result working in turn modulo each of the finitely many minimal primes of R. We may therefore assume that R is a domain. We can consequently write $R = S/P$, where S is Cohen-Macaulay. Let h be the height of P. Then we can choose $y_1, \ldots, y_h \in P$ and $z_1, \ldots, z_{k+1}$ in S as in the conclusion of the Lemma just above, i.e., so $y_1, \ldots, y_h, z_1, \ldots, z_{k+1}$ is a regular sequence in S, and so that we may replace $x_1, \ldots, x_{k+1}$ by the images of the z_i in R. Since P has height h, it is a minimal prime of $J = (y_1, \ldots, y_h)S$, and so if we localize at $S - P$, we have that

P is nilpotent modulo J. Hence, for each generator g_i of P we can choose $c_i \in S - P$ and an expoment of the form $q_i = p^{e_i}$ such that $c_i g_i^{q_i} \in J$. It follows that if $c \in S - P$ is the product of the c_i and q_0 is the maximum of the q_i, then $cP^{[q_0]} \subseteq J$.

Now suppose that we have a relation

$$rx_{k+1} = r_1x_1 + \cdots + r_kx_k$$

in R. Then we can lift r, r_1, $\ldots$, r_k to elements s, s_1, $\ldots$, $s_k \in S$ such that

$$sz_{k+1} = s_1z_1 + \cdots + s_kz_k + v,$$

where $v \in P$. Then for all $q \geq q_0$ we may raise both sides to the q th power and multiply by c to obtain

$$cs^qz_{k+1}^q = cs_1^qz_1^q + \cdots + cs_k^qz_k^q + cv^q;$$

moreover, $cv^q \in (y_1, \ldots , y_h)$. Therefore

$$cs^qz_{k+1}^q \in (z_1^q, \ldots , z_k^q, y_1, \ldots , y_h)S.$$

Since $y_1, \ldots , y_h, z_1^q, \ldots , z_{k+1}^q$ is a regular sequence in S, we have that

$$cs^q \in (z_1^q, \ldots , z_k^q)S + (y_1, \ldots , y_h)S.$$

Let $\overline{c} \in R^\circ$ be the image of c. Then, working modulo $P \supseteq (y_1, \ldots , y_h)R$, we have

$$\overline{c}r^q \in (x_1, \ldots , x_k)^{[q]}$$

for all $q \geq q_0$, and so $r \in (x_1, \ldots , x_k)^*$ in R, as required. This completes the argument for part (a).

It remains to prove part (b). Suppose that R has a test element $d \in R^\circ$, that $r \in R$, and that $rx_{k+1} \in I_k^*$. Then there exists $c \in R^\circ$ such that $c(rx_{k+1})^q \in (I_k^*)^{[q]}$ for all $q \gg 0$. Note that $(I_k^*)^{[q]} \subseteq (I_k^{[q]})^*$, so that $cr^qx_{k+1}^q \in (I_k^{[q]})^*$, and $dcr^qx_{k+1}^q \in I_k^{[q]}$. From part (a), it follows that $dcr^q \in (I_k^{[q]})^*$ for all $q \gg 0$, and so $d^2cr^q \in I_k^{[q]}$ for all $q \gg 0$. But then $r \in I_k^*$, as required. $\square$

Corollary. *Let R be a Noetherian ring of prime characteristic $p > 0$ that is a homomorphic image of a Cohen-Macaulay ring, and suppose that R is weakly F-regular. Then R is Cohen-Macaulay.*

Proof. Consider a local ring of R at a maximal ideal. Then this local ring remains weakly F-regular, and is normal. Therefore, we may assume that R is a local domain. Let $x_1, \ldots , x_n$ be a system of parameters. Then for every $k < n$, $(x_1, \ldots , x_k) :_R x_{k+1} \subseteq (x_1, \ldots , x_k)^* = (x_1, \ldots , x_k)$, since $(x_1, \ldots , x_k)$ is tightly closed. $\square$