Math 711: Lecture of October 8, 2007

Properties of regular sequences

In the sequel we shall need to make use of certain standard facts about regular sequences on a module: for convenience, we collect these facts here. Many of the proofs can be made simpler in the case of a regular sequence that is *permutable*, i.e., whose terms form a regular sequence in every order. This hypothesis holds automatically for regular sequences on a finitely generated module over a local ring. However, we shall give complete proofs here for the general case, without assuming permutability. The following fact will be needed repeatedly.

Lemma. *Let R be a ring, M an R-module, and let $x_1, \ldots, x_n$ be a possibly improper regular sequence on M. If $u_1, \ldots, u_n \in M$ are such that*

$$\sum_{j=1}^{n} x_j u_j = 0,$$

then every $u_j \in (x_1, \ldots, x_n)M$.

Proof. We use induction on n. The case where $n = 1$ is obvious. We have from the definition of possibly improper regular sequence that $u_n = \sum_{j=1}^{n-1} x_j v_j$, with $v_1, \ldots, v_{n-1} \in M$, and so $\sum_{j=1}^{n-1} x_j(u_j + x_n v_j) = 0$. By the induction hypothesis, every $u_j + x_n v_j \in (x_1, \ldots, x_{n-1})M$, from which the desired conclusion follows at once $\square$

Proposition. *Let $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_h = M$ be a finite filtration of M. If $x_1, \ldots, x_n$ is a possibly improper regular sequence on every factor M_{k+1}/M_k, $0 \leq k \leq h-1$, then it is a possibly improper regular sequence on M. If, moreover, it is a regular sequence on M/M_{h-1}, then it is a regular sequence on M.*

Proof. If we know the result in the possibly improper case, the final statement follows, for if $I = (x_1, \ldots, x_n)R$ and $IM = M$, then the same hold for every homomorphic image of M, contradicting the hypothesis on M/M_{h-1}.

It remains to prove the result when $x_1, \ldots, x_n$ is a possibly improper regular sequence on every factor. The case where $h = 1$ is obvious. We use induction on h. Suppose that $h = 2$, so that we have a short exact sequence

$$0 \to M_1 \to M \to N \to 0$$

and $x_1, \dots, x_n$ is a possibly regular sequence on M_1 and N. Then x_1 is a nonzerodivisor on M, for if $x_1 u = 0$, then x_1 kills the image of u in N. But this shows that the image of u in N must be 0, which means that $u \in M_1$. But x_1 is not a zerodivisor on M_1. It follows that

$$0 \to xM_1 \to xM \to xN \to 0$$

is also exact, since it is isomorphic with the original short exact sequence. Therefore, we have a short exact sequence of quotients

$$0 \to M_1/x_1M_1 \to M/x_1N \to M/x_1N \to 0.$$

We may now apply the induction hypothesis to conclude that $x_2, \dots, x_n$ is a possibly improper regular sequence on M/x_1M, and hence that $x_1, \dots, x_n$ is a possibly improper regular sequence on M.

We now carry through the induction on h. Suppose we know the result for filtrations of length $h-1$. We can conclude that $x_1, \dots, x_n$ is a possibly improper regular sequence on M_{h-1}, and we also have this for M/M_{h-1}. The result for M now follows from the case where $h = 2$. □

Theorem. *Let $x_1, \dots, x_n \in R$ and let M be an R-module. Let $t_1, \dots, t_n$ be integers ≥ 1. Then $x_1, \dots, x_n$ is a regular sequence (respectively, a possibly improper regular sequence) on M iff $x_1^{t_1}, \dots, x_n^{t_n}$ is a regular sequence on M (respectively, a possibly improper regular sequence on M).*

Proof. If $IM = M$ then $I^kM = M$ for all k. If each of I and J has a power in the other, it follows that $IM = M$ iff $JM = M$. Thus, we will have a proper regular sequence in one case iff we do in the other, once we have established that we have a possibly improper regular sequence. In the sequel we deal with possibly improper regular sequences, but for the rest of this proof we omit the words "possibly improper."

Suppose that $x_1, \dots, x_n$ is a regular sequence on M. By induction on n, it will suffice to show that $x_1^{t_1}, x_2, \dots, x_n$ is a regular sequence on M: we may pass to $x_2, \dots, x_n$ and $M/x_n^{t_1}M$ and then apply the induction hypothesis. It is clear that $x_1^{t_1}$ is a nonzerodivisor when x_1 is. Moreover, $M/x_1^{t_1}M$ has a finite filtration by submodules $x_1^jM/x_1^{t_1}M$ with factors $x_1^jM/x_1^{j+1}M \cong M/x_1M$, $1 \leq j \leq t_1 - 1$. Since $x_2, \dots, x_n$ is a regular sequence on each factor, it is a regular sequence on $M/x_1^{t_1}M$ by the preceding Proposition.

For the other implication, it will suffice to show that if $x_1, \dots, x_{j-1}, x_j^t, x_{j+1}, \dots, x_n$ is a regular sequence on M, then $x_1, \dots, x_n$ is: we may change the exponents to 1 one at a time. The issue may be considered mod $(x_1, \dots, x_{j-1})M$. Therefore, it suffices to consider the case $j = 1$, and we need only show that if $x_1^t, x_2, \dots, x_n$ is a regular sequence on M then so is $x_1, \dots, x_n$. It is clear that if x_1^t is a nonzerodivisor then so is x_1.

By induction on n we may assume that $x_1, \dots, x_{n-1}$ is a regular sequence on M. We need to show that if $x_nu \in (x_1, \dots, x_{n-1})M$, then $u \in (x_1, x_2, \dots, x_{n-1})M$. If we multiply by x_1^{t-1}, we find that

$$x_n(x_1^{t-1}u) \in (x_1^t, x_2, \dots, x_{n-1})M,$$

and so

$$x_1^{t-1}u = x_1^t v_1 + x_2 v_2 + \cdots + x_{n-1}v_{n-1},$$

i.e.,

$$x_1^{t-1}(u - x_1 v_1) - x_2 v_2 - \cdots - x_{n-1}v_{n-1} = 0.$$

By the induction hypothesis, $x_1, \ldots, x_{n-1}$ is a regular sequence on M, and by the first part, $x_1^{t-1}, x_2, ..., x_{n-1}$ is a regular sequence on M. By the Lemma on p. 1, we have that

$$u - x_1 v_1 \in (x_1^{t-1}, x_2, \ldots, x_{n-1})M,$$

and so $u \in (x_1, \ldots, x_{n-1})M$, as required. □

Theorem. *Let $x_1, \ldots, x_n$ be a regular sequence on the R-module M, and let I denote the ideal $(x_1, \ldots, x_n)R$. Let $a_1, \ldots, a_n$ be nonnegative integers, and suppose that u, $u_1, \ldots, u_n$ are elements of M such that*

$$(\#) \quad x_1^{a_1} \cdots x_n^{a_n} u = \sum_{j=1}^{n} x_j^{a_j+1} u_j.$$

Then $u \in IM$.

Proof. We use induction on the number of nonzero a_j: we are done if all are 0. If $a_i > 0$, let y be $\Pi_{j\neq i} x_j^{a_j}$. Rewrite (#) as $\sum_{j\neq i} x_j^{a_j+1}u_j - x_i^{a_j}(yu - x_i u_i) = 0$. Since powers of the x_j are again regular, the Lemma on p. 1 yields that $yu - x_i u_i \in x_i^{a_i}M + (x_j^{a_j+1} : j \neq i)M$ and so $yu \in x_i M + (x_j^{a_j+1} : j \neq i)M$. Now $a_i = 0$ in the monomial y, and there is one fewer nonzero a_j. The desired result now follows from the induction hypothesis. □

If I is an ideal of a ring R, we can form the *associated graded ring*

$$\mathrm{gr}_I(R) = R/I \oplus I/I^2 \oplus \cdots \oplus I^k/I^{k+1} \oplus \cdots,$$

an $\mathbb{N}$-graded ring whose kth graded piece is I^k/I^{k+1}. If $f \in I^h$ represents an element $a \in I^h/I^{h+1} = [\mathrm{gr}_I R]_h$ and $g \in I^k$ represents an element $b \in I^k/I^{k+1} = [\mathrm{gr}_I(R)]_k$, then ab is the class of fg in I^{h+k}/I^{h+k+1}. Likewise, if M is an R-module, we can form

$$\mathrm{gr}_I M = M/IM \oplus IM/I^2M \oplus \cdots \oplus I^kM/I^{k+1}M \oplus \cdots.$$

This is an $\mathbb{N}$-graded module over $\mathrm{gr}_I(R)$ in an obvious way: with f and a as above, if $u \in I^kM$ represents an element $z \in I^kM/I^{k+1}M$, then the class of fu in $I^{h+k}M/I^{h+k+1}M$ represents az.

If $x_1, \ldots, x_n \in R$ generate I, the classes $[x_i] \in I/I^2$ generate $\mathrm{gr}_I(R)$ as an (R/I)-algebra. Let $\theta : (R/I)[X_1, \ldots, X_n] \twoheadrightarrow \mathrm{gr}_I(R)$ be the (R/I)-algebra map such that $X_i \mapsto [x_i]$. This is a surjection of graded (R/I)-algebras. By restriction of scalars, $\mathrm{gr}_I(M)$ is also a module over $(R/I)[X_1, \dots, X_n]$. The (R/I)-linear map $M/IM \hookrightarrow \mathrm{gr}_I M$ then gives a map

$$\theta_M : (R/I)[X_1, \dots, X_n] \otimes_{R/I} M/IM \to \mathrm{gr}_I(M).$$

Note that $\theta_R = \theta$. If $u \in M$ represents $[u]$ in M/IM and $t_1, \dots, t_n$ are nonnegative integers whose sum is k, then

$$X_1^{t_1} \cdots X_n^{t_n} \otimes [u] \mapsto [x_1^{t_1} \cdots x_n^{t_n} u],$$

where the right hand side is to be interpreted in $I^k M/I^{k+1}M$. Note that θ_M is surjective.

Theorem. *Let $x_1, \dots, x_n$ be a regular sequence on the R-module M, and suppose that $I = (x_1, \dots, x_n)R$. Let $X_1, \dots, X_n$ be indeterminates over the ring R/I. Then*

$$\mathrm{gr}_I(M) \cong (R/I)[X_1, \dots, X_n] \otimes_{R/I} (M/IM)$$

in such a way that the action of $[x_i] \in I/I^2 = [\mathrm{gr}_I(R)]_1$ on $\mathrm{gr}_I(M)$ is the same as multiplication by the variable X_i.

In particular, if $x_1, \dots, x_n$ is a regular sequence in R, then $\mathrm{gr}_I(R) \cong (R/I)[X_1, \dots, X_n]$ in such a way that $[x_i]$ corresponds to X_i.

In other words, if $x_1, \dots, x_n$ is a regular sequence on M (respectively, R), then the map θ_M (respectively, θ) discussed in the paragraph above is an isomorphism.

Proof. The issue is whether θ_M is injective. If not, there is a nontrivial relation on the monomials in the elements $[x_i]$ with coefficients in M/IM, and then there must be such a relation that is homogeneous of, say, degree k. Lifting to M, we see that this means that there is an $(M - IM)$-linear combination of mutually distinct monomials of degree k in $x_1, \dots, x_n$ which is in $I^{k+1}M$. Choose one monomial term in this relation: it will have the form $x_1^{a_1} \cdots x_n^{a_n} u$, where the sum of the a_j is k and $u \in M - IM$. The other monomials of degree k in the elements $x_1, \dots, x_n$ and the monomial generators of I^{k+1} all have as a factor at least one of the terms $x_1^{a_1+1}, \dots, x_n^{a_n+1}$. This yields that

$$(\#) \quad (\Pi_j x_j^{a_j})u = \sum_{j=1}^{n} x_j^{a_j+1} u_j.$$

By the preceding Theorem, $u \in IM$, contradictioning that $u \in M - IM$. $\square$

Another description of the Koszul complex

Let R be a ring and let $\underline{x} = x_1, \dots, x_n \in R$. In our development of the Koszul complex, we showed that $\mathcal{K}_i(\underline{x};\, R)$ has $\binom{n}{i}$ generators $u_{j_1 \cdots j_i}$ where $1 \le j_1 < \cdots < j_i \le n$, so that the generators may the thought of as indexed by strictly increasing sequences of integers between 1 and n inclusive of length i. We may also think of the generators as indexed by the i element subsets of $\{1, \ldots, n\}$.

This means that with

$$G = \mathcal{K}_1(\underline{x};\, R) = Ku_1 \oplus \cdots \oplus Ku_n,$$

we have that

$$\mathcal{K}_i(\underline{x};\, R) \cong \bigwedge^i G,$$

for all $i \in \mathbb{Z}$ in such a way that $u_{j_1 \cdots j_i}$ corresponds to $u_{j_1} \wedge \cdots \wedge u_{j_i}$. Thus, the Koszul complex coincides with the *skew-commutative* $\mathbb{N}$-*graded* algebra $\bigwedge^\bullet(G)$. (A *skew-commutative* $\mathbb{N}$-*graded* algebra is an associative $\mathbb{N}$-graded ring with identity such that if u and v are forms of degree d, e respectively, then $vu = (-1)^{de}uv$. The elements of even degree span a subalgebra that is in the center.) A *graded derivation* of such an algebra of degree -1 is a $\mathbb{Z}$-linear map δ that lowers degrees by 1 and satisfies

$$\delta(uv) = \big(\delta(u)\big)v + (-1)^d u\delta(v)$$

when u and v or forms as above.

It is easy to check that the differential of the Koszul complex is a derivation of degree -1 in the sense specified. Moreover, given any R-linear map $G \to R$, it extends uniqely to an R-linear derivation of $\bigwedge^\bullet(G)$ of degree -1. If we choose a basis for G, call it $u_1, \ldots, u_n$, and let x_i be the value of the map on u_i, we recover $\mathcal{K}_\bullet(\underline{x};\ R)$ in this way.

Maps of quotients by regular sequences

Let $\underline{x} = x_1, \ldots, x_n$ and $\underline{y} = y_1, \ldots, y_n$ be two regular sequences in R such that $J = (y_1, \ldots, y_n)R \subseteq (x_1, \ldots, x_n)R = I$. It is obvious that there is a surjection $R/J \twoheadrightarrow R/I$. It is far less obvious, but very useful, that there is an injection $R/I \to R/J$.

Theorem. *Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be two regular sequences on a Noetherian module M over a Noetherian ring R. Suppose that*

$$J = (y_1, \ldots, y_n)R \subseteq (x_1, \ldots, x_n)R = I.$$

Choose elements $a_{ij} \in R$ such that for all j, $y_j = \sum_{i=1}^n a_{ij}x_i$. Let A be the matrix $\big(a_{ij}\big)$, so that we have a matrix equation

$$(y_1 \ \ldots \ y_n) = (x_1 \ \ldots \ x_n)A.$$

Let $D = \det(A)$. Then $DI \subseteq J$, and the map $M/IM \to M/JM$ induced by multiplication by D on the numerators in injective.

Proof. Let B be the classical adjoint of A, so that $BA = AB = DI_n$, where I_n is the $n \times n$ identity matrix. Then

$$(y_1 \, \ldots \, y_n)B = (x_1 \, \ldots \, x_n)AB = (x_1 \, \ldots \, x_n)D$$

shows that $DI \subseteq J$.

The surjection $R/J \twoheadrightarrow R/I$ lifts to a map of projective resolutions of these modules: we can use any projective resolutions, but in this case we use the two Koszul complexes $\mathcal{K}_\bullet(\underline{x};\, R)$ and $\mathcal{K}_\bullet(\underline{y};\, R)$. With these specific resolutions, we can use the matrix A to give the lifting as far as degree 1:

$$\begin{array}{ccccccc}
\mathcal{K}_1(\underline{x};\, R) & \xrightarrow{(x_1 \, \ldots \, x_n)} & R & \longrightarrow & R/(x_1, \, \ldots \, , x_n) & \longrightarrow & 0 \\
{\scriptstyle A}\Big\uparrow & & {\scriptstyle \mathbf{1}_R}\Big\uparrow & & \Big\uparrow & & \\
\mathcal{K}_1(\underline{y};\, R) & \xrightarrow{(y_1 \, \ldots \, y_n)} & R & \longrightarrow & R/(y_1, \, \ldots \, , y_n) & \longrightarrow & 0
\end{array}$$

Here, we are using the usual bases for $\mathcal{K}_1(\underline{x};\, R)$ and $\mathcal{K}_1(\underline{y};\, R)$. It is easy to check that if we use the maps

$$\bigwedge^i A : \mathcal{K}_i(\underline{y};\, R) \to \mathcal{K}_i(\underline{x};\, R)$$

for all i, we get a map of complexes. This means that the map

$$R \cong \mathcal{K}_n(\underline{y};\, R) \to \mathcal{K}_n(\underline{x};\, R) \cong R$$

is given by multiplication by D. It follows that the map induced by multiplication by D gives the induced map

$$\mathrm{Ext}^n_R(R/(x_1, \, \ldots \, , x_d), \, M) \to \mathrm{Ext}^n(R/(y_1, \, \ldots \, , y_n), \, M).$$

We have already seen that these top Ext modules may be identified with $M/(x_1, \, \ldots \, , x_{)}M$ and $M/(y_1, \, \ldots \, , y_n)M$, respectively: this is the special case of the Theorem at the bottom of p. 4 of the Lecture Notes of October 5 in the case where $i = n$.

Consider the short exact sequence

$$0 \to I/J \to R/J \to R/I \to 0.$$

The long exact sequence for Ext yields, in part,

$$\mathrm{Ext}^{n-1}_R(I/J; M) \to \mathrm{Ext}^n_R(R/I;\, M) \to \mathrm{Ext}^n_R(R/J; M).$$

Since the depth of M on $\mathrm{Ann}_R(I/J) \supseteq J$ is at least n, the leftmost term vanishes, which proves the injectivity of the map on the right. $\square$

Remark. We focus on the case where $M = R$: a similar comment may be made in general. We simply want to emphasize that the identification of $\mathrm{Ext}^n_R(R/I,\, R)$ with R/I is *not* canonical: it depends on the choice of generators for I. But a different identification can only arise from multiplication by a unit of R/I. A similar remark applies to the identification of $\mathrm{Ext}^n_R(R/J,\, R)$ with R/J.

Remark. The hypothesis that R and M be Noetherian is not really needed. Even if the ring is not Noetherian, if the annihilator of a module N contains a regular sequence $x_1,\, \dots,\, x_d$ of length d on M, it is true that $\mathrm{Ext}^i_R(N,\, M) = 0$ for $i < d$. If $d \geq 1$, it is easy to see that any map $N \to M$ must be 0: any element in the image of the map must be killed by x_1, and $\mathrm{Ann}_M x_1 = 0$. The inductive step in the argument is then the same as in the Noetherian case: consider the long exact sequence for Ext arising when $\mathrm{Hom}_R(N,\, \_\,)$ is applied to

$$0 \to M \xrightarrow{x_1 \cdot} M \to M/x_1M \to 0.$$

The type of a Cohen-Macaulay module over a local ring

Let $(R,\, m,\, K)$ be local and let M be a finitely generated nonzero R-module that is Cohen-Macaulay, i.e., every system of parameters for R/I, where $I = \mathrm{Ann}_R M$, is a regular sequence on M. (It is equivalent to assume that $\mathrm{depth}_m M = \dim(M)$.) Recall that the *socle* of an R-module M is $\mathrm{Ann}_M m \cong \mathrm{Hom}_R(K,\, M)$. It turns out that for any maximal regular sequence $x_1,\, \dots,\, x_d$ on M, the dimension as a K-vector space of the socle in $M/(x_1,\, \dots,\, x_d)M$ is independent of the choice of the system of parameters. One way to see this is as follows:

Propostion. *Let $(R,\, m,\, K)$ and M be as above with M Cohen-Macaulay of dimension d over R. Then for every maximal regular sequence $x_1,\, \dots,\, x_d$ on M and for every i, $1 \leq i \leq d$,*

$$\mathrm{Ext}^d_R(K,\, M) \cong \mathrm{Ext}^{d-i}_R\big(K,\, M/(x_1,\, \dots,\, x_i)M\big).$$

In particular, for every maximal regular sequence on M, the socle in $M/(x_1,\, \dots,\, x_d)M$ is isomorphic to $\mathrm{Ext}^d_R(K,\, M)$, and so its K-vector space dimension is independent of the choice maximal regular sequence.

Proof. The statement in the second paragraph follows from the result of the first paragraph in the case where $i = d$. By induction, the proof that

$$\mathrm{Ext}^d_R(K,\, M) \cong \mathrm{Ext}^{d-i}_R\big(K,\, M/(x_1,\, \dots,\, x_i)M\big)$$

reduces at once to the case where $i = 1$. To see this, apply the long exact sequence for Ext arising from the application of $\mathrm{Hom}_R(K,\, \_\,)$ to the short exact sequence

$$0 \to M \to M \to M/x_1M \to 0.$$

Note that $\mathrm{Ext}^j(K,\,M) = 0$ for $j < d$, since the depth of M on $\mathrm{Ann}_R K = m$ is d, and that $\mathrm{Ext}^j(K,\,M/x_1M) = 0$ for $j < d-1$, similarly. Hence, we obtain, in part,

$$0 \to \mathrm{Ext}^{d-1}_R(K,\,M/x_1M) \to \mathrm{Ext}^d_R(K,\,M) \xrightarrow{x_1\cdot} \mathrm{Ext}^d_R(K,\,M).$$

Since $x_1 \in m$ kills K, the map on the right is 0, which gives the required isomorphism. $\square$

Proposition. *Let $M \neq 0$ be a Cohen-Macaulay module over a local ring R. Let $x_1,\,\ldots\,,x_n$ and $y_1,\,\ldots\,,y_n$ be two systems of parameters on M with $(y_1,\,\ldots\,,y_n)R \subseteq (x_1,\,\ldots\,,x_n)R$ and let $A = \big(a_{ij}\big)$ be a matrix of elements of R such that $(y_1\,\ldots\,y_n) = (x_1\,\ldots\,x_n)A$. Let $D = \det(A)$. Then the map $M/(x_1,\,\ldots\,,x_n)M \to M/(y_1,\,\ldots\,,y_n)M$ induced by multiplication by D on the numerators carries the socle of $M/(x_1,\,\ldots\,,x_n)M$ isomorphically onto the socle of $M/(y_1,\,\ldots\,,y_n)M$.*

In particular, if $y_i = x_i^t$, $1 \leq i \leq n$, then the map induced by multiplication by $x_1^{t-1}\cdots x_n^{t-1}$ carries the socle of the quotient module $M/(x_1,\,\ldots\,,x_n)M$ isomorphically onto the socle of $M/(x_1^t,\,\ldots\,,x_n^t)M$.

Proof. By the Theorem on p. 5, multiplication by D gives an injection

$$M/(x_1,\,\ldots\,,x_n)M \hookrightarrow M/(y_1,\,\ldots\,,y_n)M$$

which must map the socle in the left hand module injectively into the socle in the right hand module. Since, by the preceding Proposition, the two socles have the same finite dimension as vector spaces over K, the map yields an isomorphism of the two socles. The final statement follows because in the case of this specific pair of systems of parameters, we may take A to be the diagonal matrix with diagonal entries $x_1^{t-1},\,\ldots\,,x_n^{t-1}$. $\square$

F-rational rings

Definition: F-rational rings. We shall say that a local ring $(R,\,m,\,K)$ is *F-rational* if it is a homomorphic image of a Cohen-Macaulay ring and every ideal generated by a system of parameters is tightly closed.

We first note:

Theorem. *An F-rational local ring is Cohen-Macaulay, and every ideal generated by part of a system parameters is tightly closed. Hence, an F-rational local ring is a normal domain.*

Proof. Let $x_1,\,\ldots\,,x_k$ be part of a system of parameters (it may be the empty sequence) and let $I = (x_1,\,\ldots\,,x_k)$. Let $x_1,\,\ldots\,,x_n$ be a system of parameters for R, and for every $t \geq 1$ let $J_t = (x_1,\,\ldots\,,x_k,\,x_{k+1}^t,\,\ldots\,,x_n^t)R$. Then for all t, $I \subseteq J_t$ and J_t is tightly closed, so that $I^* \subseteq J_t$ and $I^* \subseteq \bigcap_t J_t = I$, as required. In particular, (0) and principal ideals generated by nonzerodivisors are tightly closed, so that R is a normal domain, by the

Theorem on the top of p. 5 of the Lecture Notes from September 17. In particular, R is equidimensional, and by part (a) of the Theorem on colon-capturing from p. 9 of the Lecture Notes from October 5, we have that for every k, $0 \leq k \leq n-1$,

$$(x_1, \ldots, x_k) :_R x_{k+1} \subseteq (x_1, \ldots, x_k)^* = (x_1, \ldots, x_k),$$

so that R is Cohen-Macaulay. $\square$

Theorem. *Let (R, m, K) be a reduced local ring of prime characteristic $p > 0$. If R is Cohen-Macaulay and the ideal $I = (x_1, \ldots, x_n)$ generated by one system of parameters is tightly closed, then R is F-rational, i.e., every ideal generated by part of a system of parameters is tightly closed.*

Proof. Let $I_t = (x_1^t, \ldots, x_n^t)R$. We first show that all of the ideals I_t are tightly closed. If not, suppose that $u \in (I_t)^* - I_t$. Since $(I_t)^*/I_t$ has finite length, u has a nonzero multiple v that represents an element of the socle of I_t^*/I_t, which is contained in the socle of R/I_t. Thus, we might as well assume that $u = v$ represents an element of the socle in R/I_t. By the last statement of the Proposition on p. 8, , we can choose z representing an element of the socle in R/I such that the class of v mod I has the form $[x_1^{t-1} \cdots x_n^{t-1} z]$. Then $x_1^{t-1} \cdots x_n^{t-1} z$ also represents an element of $I^* - I$. Hence, we can choose $c \in R^\circ$ such that for all $q \gg 0$,

$$c(x_1^{t-1} \cdots x_n^{t-1} z)^q \in I_t^{[q]} = I_{tq},$$

i.e., $cx_1^{tq-q} \cdots x_n^{tq-q} z^q \in I_{tq}$, which implies that

$$cz^q \in \big((x_1^q)^t, \ldots, (x_n^q)^t\big) :_R (x_1^q)^{t-1} \cdots (x_n^q)^{t-1}.$$

By the Theorem on p. 3 applied to the regular sequence $x_1^q, \ldots, x_n^q$, the right hand side is $(x_1^q, \ldots, x_n^q) = I^{[q]}$, and so

$$cz^q \in I^{[q]}$$

for all $q \gg 0$. This shows that $z \in I^* = I$, contradicting the fact that z represents a nonzero socle element in R/I.

Now consider any system of parameters $y_1, \ldots, y_n$. For $t \gg 0$, $(x_1^t, \ldots, x_n^t)R \subseteq (y_1, \ldots, y_n)R$. Then there is an injectiion $R/y_1, \ldots, y_n)R \hookrightarrow R/(x_1^t, \ldots, x_n^t)R$ by the Theorem at the bottom of p. 5. Since 0 is tightly closed in the latter, it is tightly closed in $R/(y_1, \ldots, y_n)R$, and so $(y_1, \ldots, y_n)R$ is tightly closed in R. $\square$

We shall see soon that under mild conditions, if (R, m, K) is a local ring of prime characteristic $p > 0$ and a single ideal generated by a system of parameters is tightly closed, then R is F-rational: we can prove that R is Cohen-Macaulay even though we are not assuming it.