Math 711: Lecture of October 10, 2007

We now want to make precise the assertion at the end of the preceding lecture to the effect that, under mild conditions on the local ring R, if one system of parameters of R generates a tightly closed ideal then R is F-rational. We already know that this is true when R is Cohen-Macaulay. The new point is that we do not need to assume that R is Cohen-Macaulay — we can prove it. However, we need the strong form of colon-capturing, and so we assume the existence of a test element. The following Theorem will enable us to prove the result that we want.

Theorem. *Let (R, m, K) be a reduced local ring of prime characteristic $p > 0$, and let $x_1, \ldots, x_n$ be a sequence of elements of m such that $I_k = (x_1, \ldots, x_k)$ has height k modulo every minimal prime of R, $1 \leq k \leq n$. Suppose that R has a test element. If $(x_1, \ldots, x_n)R$ is tighly closed, then I_k is tightly closed, $0 \leq k \leq n$, and $x_1, \ldots, x_n$ is a regular sequence in R.*

Proof. We first prove that every I_k is tightly closed, $0 \leq k \leq n$, by reverse induction on k. We are given that I_n is tightly closed. Now suppose that we know that I_{k+1} is tightly closed, where $0 \leq k \leq n-1$. We prove that I_k is tightly closed. Let $u \in I_k^*$ be given. Since $I_k \subseteq I_{k+1}$, we have that $I_k^* \subseteq I_{k+1}^* = I_{k+1}$ by hypothesis,, and $I_{k+1} = I_k + x_{k+1}R$. Thus, $u = v + x_{k+1}w$, where $v \in I_k$ and $w \in R$. But then $x_{k+1}w = u - v \in I_k^*$, since $u \in I_k^*$ and $v \in I_k$. Consequently, $v \in I_k^* :_R x_{k+1}$. By part (b) of the Theorem on colon-capturing at the bottom of p. 9 of the Lecture Notes from October 5, we have that $I_k^* :_R x_{k+1} = I_k^*$. That is, $u \in I_k + x_{k+1}I_k^*$. Since $u \in I_k^*$ was arbitrary, we have shown that $I_k^* \subseteq I_k + x_{k+1}I_k^*$, and the opposite inclusion is obvious. Let $N = I_k^*/I_k$. Then we have that $x_{k+1}N = N$, and so $N = 0$ by Nakayama's Lemma. But this says that $I_k^* = I_k$, as required. The fact that the x_i form a regular sequence is then obvious from the Theorem at the bottom of p. 9 of the Lecture Notes from October 5 cited above. $\square$

We then have:

Theorem. *Let (R, m, K) be a reduced, equidimensional local ring that is a homomorphic image of a Cohen-Macaulay ring. Suppose that R has a test element. If the ideal generated by one system of parameters of R is tightly closed, then R is F-rational. That is, R is Cohen-Macaulay and every ideal generated by part of a system of parameters is tightly closed.*

Proof. By the preceding Theorem, R is Cohen-Macaulay, and the result now follows from the Theorem on p. 9 of the Lecture Notes from October 8. $\square$

A localization of an F-rational local ring at any prime is F-rational. The proof is left as an exercise in Problem Set #3. It is therefore natural to define a Noetherian ring R of prime characteristic $p > 0$ to be *F-rational* if its localization at every maximal ideal (equivalently, at every prime ideal) is F-rational.

Definition: Gorenstein local rings. A local ring $(R,\, m,\, K)$ is called *Gorenstein* if it is Cohen-Macaulay of type 1. Thus, if $x_1,\, \ldots\, , x_n$ is any system of parameters in R, the Artin local ring $R/(x_1,\, \ldots\, , x_n)R$ has a one-dimensional socle, which is contained in every nonzero ideal of R. Notice that if $(R,\, m,\, K)$ is Gorenstein of dimension n, we know that $\mathrm{Ext}^i_R(K,\, R) = 0$ if $i < n$, while $\mathrm{Ext}^n_R(K,\, R) \cong K$.

Note that killing part of a system of parameters in a Gorenstein local ring does not change its type: hence such a quotient is again Gorenstein. Regular local rings are Gorenstein, since the quotient of a regular local ring by a regular system of parameters is K. The quotient of a regular local ring by part of a system of parameters is therefore Gorenstein. In particular, the quotient R/f of a regular local ring R by a nonzero proper principal ideal fR is Gorenstein. Such a ring R/fR is called a *local hypersurface*.

We shall need the following very important result.

Theorem. *If an Artin local ring is Gorenstein, it is injective as a module over itself.*

This was proved in seminar. See also, for example, [W. Bruns and J. Herzog, *Cohen-Macaulay rings,* Cambridge Studies in Advanced Math. **39** Cambridge Univ. Press, Cambridge, 1993] Theorems (3.1.17) and (3.2.10).

In consequence, we are able to prove the following most useful result. Notice that it reduces checking whether a given Gorenstein local ring is weakly F-regular to determining whether one specific element is in the tight closure of the ideal generated by one system of parameters.

Theorem. *Let $(R,\, m,\, K)$ be a reduced Gorenstein local ring of prime characteristic $p > 0$. Let $x_1,\, \ldots\, , x_n$ be a system of parameters for R, and let u in R represent a generator of the socle in $R/(x_1,\, \ldots\, , x_n)R$. Then the following conditions are equivalent.*

(1) *R is weakly F-regular.*

(2) *R is F-rational*

(3) *$(x_1,\, \ldots\, , x_n)R$ is tightly closed.*

(4) *The element u is not in the tight closure of $(x_1,\, \ldots\, , x_n)R$.*

Proof. Let $I = (x_1, \ldots, x_n)$. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). But (4) $\Rightarrow$ (3) because if I^* is strictly larger than I, then I^*/I is a nonzero ideal of R/I and must contain the socle element represented by u, from which it follows that $u \in I^*$. The fact the (3) $\Rightarrow$ (2) follows from the Theorem on p. 9 of the Lecture Notes from October 8. What remains to be proved is the most interesting implication, that (2) $\Rightarrow$ (1).

Assume that R is F-rational, and let $N \subseteq M$ be finitely generated R-modules. We must show that N is tightly closed. If not, choose $u \in N^* - M$. We may replace N by a submodule N' of M with $N \subseteq N' \subseteq M$ such that N' is maximal witth respect to the property of not containing u. We will still have that u is in the tight closure of N' in M, and $u \notin N'$. We may then replace M and N' by M/N' and 0, respectively, and u by its image in M/N'. The maximality of N' implies that u is in every submodules of M/N'. We change notation: we may assume that $u \in M$ is in every nonzero submodule of M, and that $u \in 0^*_M$.

By the Lemma on the first page of the Lecture Notes from September 17, we have that M has finite length and is killed by a power of the maximal ideal of R. Moreover, u is in every nonzero submodule of M. Let $x_1, \ldots, x_n$ be a system of parameters for R. For $t \gg 0$, we have that every x_i^t kills M. Thus, we may think of M as a module over the Artin local ring $A = R/(x_1^t, \ldots, x_n^t)R$, which is a Gorenstein Artin local ring. Let v be the socle element in A. Then

$$Ru \cong K \cong Rv \subseteq A$$

gives an injective map of $Ru \subseteq M$ to A. Since A is injective as an A-module and M is an A-module, this map extends to a map $\theta : M \to A$ that is A-linear and, hence, R-linear. We claim that θ is injective: if the kernel were nonzero, it would be a nonzero submodule of M, and so it would contain u, contradicting the fact that u has nonzero image in A. Since $M \hookrightarrow A\text{R}$, to show that 0 is tightly closed in M over R, it suffices to show that 0 is tightly closed in A over R. Since $A = R/(x_1^t, \ldots, x_n^t)R$, this is simply equivalent to the statement that $(x_1^t, \ldots, x_n^t)R$ is tightly closed in R. $\square$