Math 711: Lecture of October 17, 2007

We next want to prove the Theorem stated at the end of the Lecture Notes from October 12. Recall that $A = K[[x_1, \ldots, x_n]]$ and that Γ is cofinite in a fixed p-base Λ for K.

First note that it is clear that $K[[x_1, \ldots, x_n]] \to K_e[[x_1, \ldots, x_n]]$ is faithfully flat: every system of parameters in the former maps to a system of parameters in the extension ring, and since the extension is regular it is Cohen-Macaulay. Faithful flatness follows from the Theorem at the top of p. 2 of the Lecture Notes of September 14. Since a direct limit of flat extensions is flat, it is clear that A^Γ is flat over A.

Since $(K_e)^q \subseteq K$, we have that

$$(A^\Gamma)^q \in (K_e)^q[[x_1^q, \ldots, x_n^q]] \in K[[x_1, \ldots, x_n]] = A.$$

Thus, every $A_e = K_e[[x_1, \ldots, x_n]]$ is purely inseparable over A, and it follows that the union A^Γ is as well. Hence, $A \to A^\Gamma$ is local. Note that the maximal ideal in each A_e is $mA_e = (x_1, \ldots, x_n)A_e$. Every element of the maximal ideal of A^Γ is in the maximal ideal of some A_e, and so in $mA_e \subseteq mA^\Gamma$. Thus, $mA^\Gamma = (x_1, \ldots, x_n)A^\Gamma$ is the maximal ideal of A^Γ. The residue class field of A^Γ is clearly the direct limit of the residue class fields K_e, which is the union $\bigcup_e K_e = K^\Gamma$: this is the gamma construction applied to $A = K$.

We next want to check that A^Γ is Noetherian. Note that A^Γ is contained in the regular ring $K^\Gamma[[x_1, \ldots, x_n]] = B$, and that each of the maps $A_e \to B$ is faithfully flat. Hence, for every ideal I of A_e, $IB \cap A_e = I$. The Noetherian property for A^Γ now follows from:

Lemma. *Let $\{A_i\}_i$ be a directed family of rings and injective homomorphisms whose direct limit A embeds in a ring B. Suppose that for all i and for every ideal J of any A_i, $JB \cap A_i = I$. Then for every ideal I of A, $IB \cap A = I$. Hence, if B is Noetherian, then A is Noetherian.*

Proof. Suppose that $u \in A$, $I \subseteq A$ and $u \in IB - IA$. Then $u = f_1b_1 + \cdots + f_nb_n$ where $f_1, \ldots, f_n \in I$ and $b_1, \ldots, b_n \in B$. We can choose i so large that $u, f_1, \ldots, f_n \in A_i$, and let $J = (f_1, \ldots, f_n)A_i$. Evidently, $u \in JB \cap A_i = J$, and, clearly, $J \subseteq IA$, a contradiction.

For the final statement, let I be any ideal of A. Then a finite subset $g_1, \ldots, g_n \in I$ generates IB. Let $I_0 = (g_1, \ldots, g_n)A$. Then $I \subseteq IB \cap A = I_0B \cap A = I_0 \subseteq I$, so that $I = I_0$. $\square$

Since A^Γ is Noetherian of Krull dimension n with maximal ideal $(x_1, \ldots, x_n)A^\Gamma$, we have that A^Γ is regular. To complete the proof of the final Theorem stated in the Lecture Notes from October 12, it remains only to prove:

Theorem. *A^Γ is F-finite.*

Proof. Throughout this argument, we write K_e for $K_e^\Gamma = K[\lambda^{1/q} : \lambda \in \Gamma]$, and A_e for $K_e[[x_1, \dots, x_n]]$. Let $\theta_1, \dots, \theta_h$ be the finitely many elements that are in the p-base Λ but not in Γ. Let $\mathcal{M}$ be the set of monomials in $\theta_1^{1/p}, \dots, \theta_h^{1/p}$ of degree at most $p-1$ in each element, and let $\mathcal{N}$ be the set of monomials in $x_1^{1/p}, \dots, x_d^{1/p}$ of degree at most $p-1$ in each element. Let

$$\mathcal{T} = \mathcal{M}\mathcal{N} = \{\mu\nu : \mu \in \mathcal{M},\, \nu \in \mathcal{N}\}.$$

We shall complete the proof by showing that $\mathcal{T}$ spans $(A^\Gamma)^{1/p}$ as an A^Γ-module. First note that

$$(A^\Gamma)^{1/p} = \bigcup_e (A_e)^{1/p},$$

and for every e,

$$(A_e)^{1/p} = K_e^{1/p}[[x_1^{1/p}, \dots, x_d^{1/p}]].$$

This is spanned over $K_e^{1/p}[[x_1, \dots, x_d]]$ by $\mathcal{N}$. Also observe that $K_e^{1/p}$ is spanned over K by products of monomials in $\mathcal{N}$ and monomials in the elements $\lambda^{1/qp}$ for $\lambda \in \Gamma$, and the latter are in K_{e+1}. Hence, $K_e^{1/p}$ is spanned by $\mathcal{N}$ over K_{e+1}, and it follows that $K_e^{1/p}[[x_1, \dots, x_n]]$ is spanned by $\mathcal{N}$ over $K_{e+1}[[x_1, \dots, x_n]] = A_{e+1}$. Hence, $A_e^{1/p}$ is spanned by $\mathcal{T} = \mathcal{M}\mathcal{N}$ over A_{e+1}, as claimed. $\square$

Note that $A^\Gamma \subseteq K^\Gamma[[x_1, \dots, x_n]]$, but these are not, in general, the same. Any single power series in A^Γ has all coefficients in a single K_e. When the chain of fields K_e is infinite, we can choose $c_e \in K_{e+1} - K_e$ for every for every $e \geq 0$, and then

$$\sum_{e=0}^{\infty} c_e x^e \in K^\Gamma[[x]] - K[[x]]^\Gamma.$$

Complete tensor products, and an alternative view of the gamma construction

Let (R, m, K) be a complete local ring with coefficient field $K \subseteq R$. When $R = A = K[[x_1, \dots, x_n]]$, we may enlarge the residue class field K of A to L by considering instead $L[[x_1, \dots, x_n]]$. This construction can be done in a more functorial way, and one does not need the ring to be regular.

Consider first the ring $R_L = L \otimes_K R$. This ring need not be Noetherian, and will not be complete except in special cases, e.g., if L is finite algebraic over K. However, $R_L/mR_L \cong L$, so that mR_L is a maximal ideal of this ring, and we may form the (mR_L)-adic completion of R_L. This ring is denoted $L\widehat{\otimes}_K R$, and is called the *complete tensor product* of L with R over K. Of course, we have a map $R \to R_L \to L\widehat{\otimes}_K R$.

Note that

$$L\widehat{\otimes}_K R = \varprojlim_t \frac{L \otimes_K R}{m^t(L \otimes_K R)} \cong \varprojlim_t \big(L \otimes_K \frac{R}{m^t}\big).$$

In case $R = K[[\underline{x}]]$, where $\underline{x} = x_1, \ldots, x_n$ are formal power series indeterminates, this yields

$$\varprojlim_t L \otimes_K \big(\frac{K[[\underline{x}]]}{(\underline{x})^t}\big) \cong \varprojlim_t L \otimes_K \big(\frac{K[\underline{x}]}{(\underline{x})^t}\big) \cong \varprojlim_t \frac{L[\underline{x}]}{(\underline{x})^t} \cong L[[\underline{x}]],$$

which gives the result we wanted.

Now suppose that we have a local map $(R, m, K) \to (S, \mathfrak{n}, K)$ of complete local rings such that S is module-finite over R, i.e., over the image of R: we are not assuming that the map is injective. For every t, we have a map $R/m^tR \to S/m^tS$ and hence a map $L \otimes_K R/m^tR \to L \otimes_K S/m^tS$. This yields a map

$$(*) \quad \varprojlim_t L \otimes_K R/m^tR \to \varprojlim_t L \otimes_K S/m^tS.$$

The map $R/mS \to S/mS$ is module-finite, which shows that S/mS has Krull dimension 0. It follows that mR is primary to $\mathfrak{n}$, so that the ideals m^tR are cofinal with the power of $\mathfrak{n}$. Therefore the inverse limit on the right in $(*)$ is the same as $\varprojlim_t L \otimes_K R/\mathfrak{n}^tR$, and we see that we have a map $L\widehat{\otimes}_K R \to L\widehat{\otimes}_K S$.

We next note that when $R \to S$ is surjective, so is the map $L\widehat{\otimes}_K R \to L\widehat{\otimes}_K S$. First note that $R_L \to S_L$ is surjective, and that mR_L maps onto $\mathfrak{n}S_L$. Second, each element σ of the completion of S_L with respect to $\mathfrak{n}$ can be thought of as arising from the classes modulo successive powers of $\mathfrak{n}$ of the partial sums of a series

$$s_0 + s_1 + \cdots + s_t + \cdots$$

such that $s_t \in \mathfrak{n}^t S_L = m^t S_L$ for all $t \in \mathbb{N}$. Since m^tR_L maps onto $\mathfrak{n}^tS_L$, we can left this series to

$$r_0 + r_1 + \cdots + r_t + \cdots$$

where for every $t \in \mathbb{N}$, $r_t \in m^tR_L$ and maps to s_t. The lifted series represents an element of the completion of R_L that maps to σ.

Since every complete local ring R with coefficient field K is a homomorphic image of a ring of the form $K[[x_1, \ldots, x_n]$, it follows that $L\widehat{\otimes}_K R$ is a homomorphic image of a ring of the form $L[[x_1, \ldots, x_n]]$, and so $L\widehat{\otimes}_K R$ is a complete local ring with coefficient field L.

Next note that when $R \to S$ is a module-finite (not necessarily injective) K-homomorphism of local rings with coefficient field K, we have a map

$$(L\widehat{\otimes}_K R) \otimes_R S \to L\widehat{\otimes}_K S,$$

since both factors in the (ordinary) tensor product on the left map to $L\widehat{\otimes}_K S$. We claim that this map is an isomorphism. Since, as noted above, mS is primary to $\mathfrak{n}$, and both sides are complete in the m-adic topology, it suffices to show that the map induces an isomorphism modulo the expansions of m^t for every $t \in \mathbb{N}$. But the left hand side becomes

$$\big(L \otimes_K (R/m^t)\big) \otimes_R S \cong L \otimes_K (S/m^t S),$$

which is exactly what we need.

It follows that $L \widehat{\otimes}_K R$ is ffaithfully flat over R: we can represent R as a module-finite extension of a complete regular local ring A with the same residue class field, and then $L \widehat{\otimes}_K R = (L \widehat{\otimes}_K A) \otimes_A R$, so that the result follows from the fact that $L \widehat{\otimes} A$ is faithfully flat over A.

With this machinery available, we can construct R^Γ, when R is complete local with coefficient field K and Γ is cofinite in a p-base Λ for K, as $\bigcup_e K_e \widehat{\otimes}_K R$. If R is regular this agrees with our previous construction.

If A, R are complete local both with coefficient field K, and $A \to R$ is a local K-algebra homomorphism that is module-finite (not necessarily injective), then we have

$$K_e \widehat{\otimes}_K R = (K_e \widehat{\otimes}_K A) \otimes_A R$$

for all e. Since tensor commutes with direct limit, it follows that

$$R^\Gamma \cong A^\Gamma \otimes_A R.$$

In particular, this holds when A is regular. It follows that R^Γ is faithfully flat over R.

Properties preserved for small choices of Γ

Suppose that Λ is a p-basse for a field K of characteristic $p > 0$. We shall say that a property holds *for all sufficiently small cofinite* $\Gamma \subseteq \Lambda$ or *for all* $\Gamma \ll \Lambda$ if there exists $\Gamma_0 \subseteq \Lambda$, cofinite in Λ, such that the property holds for all $\Gamma \subseteq \Gamma_0$ that are cofinite in Λ.

We are aiming to prove the following:

Theorem. *Let B be a complete local ring of prime characteristic $p > 0$ with cooefficient field K, let Λ be a p-base for K, and and let R an algebra essentially of finite type over B. For Γ cofinite in Λ, let R^Γ denote $B^\Gamma \otimes_B R$.*

(a) *If R is a domain, then R^Γ is a domain for all $\Gamma \ll \Lambda$.*

(b) *If R is reduced, then R^Γ is reduced for all $\Gamma \ll \Lambda$.*

(c) *If $P \subseteq R$ is prime, then PR^Γ is prime for all $\Gamma \ll \Lambda$.*

(d) *If $I \subseteq R$ is radical, then IR^Γ is radical for all $\Gamma \ll \Lambda$.*

We shall also prove similar results about the behavior of the singular locus. We first note:

Lemma. *Let M be an R-module, let $P_1, \ldots, P_h$ be submodules of M, and let S be a flat R-module. Then the intersection of the submodules $S \otimes_R P_i$ for $1 \le i \le h$ is*

$$(P_1 \cap \cdots \cap P_h) \otimes_R M.$$

Here, for $P \subseteq M$, we are identifying $S \otimes_R P$ with its image in $S \otimes_R M$: of course, the map $S \otimes_R P \to S \otimes_R M$ is injective.

Proof. By a straightforward indcution on h, this comes down to the intersection of two submodules P and Q of the R-module M. We have an exact sequence

$$0 \longrightarrow P \cap Q \longrightarrow M \xrightarrow{f} (M/P \oplus M/Q)$$

where the rightmost map f sends $u \in M$ to $(u + P) \oplus (u + Q)$. Since S is R-flat, applying $S \otimes_R \_$ yields an exact sequence

$$0 \longrightarrow S \otimes_R (P \cap Q) \longrightarrow S \otimes_R M \xrightarrow{\mathbf{1}_S \otimes f} \bigl(S \otimes_R (M/P)\bigr) \oplus \bigl(S \otimes_R (M/Q)\bigr).$$

The rightmost term may be identified with

$$(S \otimes_R M)/(S \otimes_R P) \oplus (S \otimes_R M)/(S \otimes_R Q),$$

from which it follows that the kernel of $\mathbf{1}_S \otimes f$ is the intersection of $S \otimes_R P$ and $S \otimes_R Q$. Consequently, this intersection is given by $S \otimes_R (P \cap Q)$. $\square$

We next want to show that part (a) of the Theorem stated above implies the other parts.

Proof that part (a) implies the other parts of the Theorem. Part (c) follows from part (a) applied to (R/P), since

$$(R/P)^\Gamma = B^\Gamma \otimes_B (R/P) \cong R^\Gamma/PR^\Gamma.$$

To prove that $(a) \Rightarrow (d)$, let $I = P_1 \cap \cdots \cap P_n$ be the primary decomposition of the radical ideal I, where the P_i are prime. Since B^Γ is flat over B, R^Γ is flat over R. Hence, IR^Γ, which may be identified with $R^\Gamma \otimes_R I$, is the intersection of the ideals $R^\Gamma \otimes_R P_i$, $1 \le i \le h$, by the Lemma above. By part (a), we can choose Γ cofinite in Λ such that every $R^\Gamma \otimes_R P_i$ is prime, and for this Γ, IR^Γ is radical.

Finally, (c) is part (d) in the case where $I = (0)$. $\square$

It remains to prove part (a). Several preliminary results are needed. We begin by replacing B by its image in the domain R, taking the image of K as a coefficient ring. Thus, we may assume that $B \hookrightarrow R$ is injective. Then B is a module-finite extension of a subring of the form $K[[x_1, \ldots, x_n]]$ with the same coefficient field, by the structure theory of complete local rings. We still have that R is essentially of finite type over A. Moreover, $B^\Gamma \cong A^\Gamma \otimes_A B$, from which it follows that $R^\Gamma \cong A^\Gamma \otimes_R A$. Therefore, in proving part (a) of the Theorem, it suffices to consider the case where $B = A = K[[x_1, \ldots, x_n]]$ and $A \subseteq R$. For each Γ cofinite in $\Lambda \subseteq K$, let $\mathcal{L}_\Gamma$ denote the fraction field of A_Γ. Let $\mathcal{L}$ denote the fraction field of A. Let Ω be any field finitely generated over R that contains the fraction field of R. To prove part (a) of the Theorem stated on p. 4, it will suffice to prove the following:

Theorem. *Let K be a field of characteristic p with p-base Λ. Let $A = K[[x_1, \dots, x_n]]$, and let $\mathcal{L}$, A^Γ and $\mathcal{L}_\Gamma$ be defined as above for every cofinite subset Γ of Λ. Let Ω be any field finitely generated over $\mathcal{L}$. Then for all $\Gamma \ll \Lambda$, $\mathcal{L}_\Gamma \otimes_{\mathcal{L}} \Omega$ is a field.*

We postpone the proof of this result. We first want to see (just below) that it implies part (a) of the Theorem stated on p. 4. Beyond that, we shall need to prove some auxiliary results first.

To see why the preceding Theorem implies part (a) of the Theorem on page 4, choose Ω containing the fraction field of R (we can choose $\Omega = \mathrm{frac}\,(R)$, for example). Since A^Γ is A-flat, we have an injection $A^\Gamma \otimes_A R \hookrightarrow A^\Gamma \otimes_A \Omega$. Thus, it suffices to show that this ring is a domain. Since the elements of $A - \{0\}$ are already invertible in Ω, we have that $\Omega \cong \mathrm{frac}\,(A) \otimes_A \Omega$. Since A^Γ is purely inseparable over A, inverting the nonzero elements of A inverts all nonzero elements of A^Γ. Moreover, the tensor product of two $\mathrm{frac}\,(A)$-modules over $\mathrm{frac}\,(A)$ is the same as their tensor product over A. Hence,

$$A^\Gamma \otimes_A \Omega \cong A^\Gamma \otimes_A \mathrm{frac}\,(A) \otimes_A \Omega \cong \mathrm{frac}\,(A^\Gamma) \otimes_{\mathrm{frac}(A)} \Omega = \mathcal{L}_\Gamma \otimes_{\mathcal{L}} \Omega.$$

It is now clear that Theorem above implies part (a) of the Theorem on p. 4.

In order to prove the Theorem above, we need several preliminary results. One of them is quite easy:

Lemma. *Let K be a field of characteristic $p > 0$ and let Λ be a p-base for K. The family of subfields K^Γ as Γ runs through the cofinite subsets of Λ is directed by $\supseteq$, and the intersection of these fields is K.*

Proof. K^∞ has as a basis 1 and all monomials

$$(\#) \quad \lambda_1^{\alpha_1} \cdots \lambda_t^{\alpha_t}$$

where t is some positive integer, $\lambda_1, \dots, \lambda_t$ are mutually distinct elements of Λ, and the α_j are positive rational numbers in $(0, 1)$ whose denominators are powers of p. If u were in the intersection and not in K it would have a unique representation as a K-linear combination of these elements, including at least one monomial μ as above other than 1. Choose $\lambda \in \Lambda$ that occurs in the monomial μ with positive exponent. Choose Γ cofinite in Λ such that $\lambda \notin \Gamma$. Then the monomial μ is not in K^Γ, which has a basis consisting of 1 and all monomials as in (#) such that the λ_j occurring are in Γ. It follows that $u \notin K^\Gamma$. $\square$

We shall also need the following result, as well as part (b) of the Theorem stated after it.

Theorem. *Let $\mathcal{L}$ be a field of characteristic $p > 0$, and let $\mathcal{L}'$ be a finite purely inseparable extension of $\mathcal{L}$. Let $\{\mathcal{L}_i\}_i$ be a family of fields directed by $\supseteq$ whose intersection is $\mathcal{L}$. Then there exists j such that for all $i \leq j$, $\mathcal{L}_i \otimes \mathcal{L}'$ is a field.*

Theorem. *Let* $\{\mathcal{K}_i\}_i$ *be a nonempty family of subfields of an ambient field* $\mathcal{K}_0$ *such that the family is directed by* $\supseteq$*, and has intersection* $\mathcal{K}$*. Let* $x_1, \dots, x_n$ *be formal power series indeterminates over these fields. Then*

(a) $\bigcap_i \operatorname{frac}(\mathcal{K}_i[x_1, \dots, x_n]) = \operatorname{frac}(\mathcal{K}[x_1, \dots, x_n])$.

(b) $\bigcap_i \operatorname{frac}(\mathcal{K}_i[[x_1, \dots, x_n]]) = \operatorname{frac}(\mathcal{K}[[x_1, \dots, x_n]])$.

We note that part (a) is easy. Choose an arbitrary total ordering of the monomials in the variables $x_1, \dots, x_n$. Let f/g be an element of the intersection on the left hand side written as the ratio of polynomials $f,\ g \neq 0$ in $\mathcal{K}_0[x_1, \dots, x_n]$, where f and g are chosen so that $\mathrm{GCD}(f,\ g) = 1$. Also choose g so that the greatest monomial occurring has coefficient 1. This representation is unique. If the same element is also in $\operatorname{frac}(\mathcal{K}_i[x_1, \dots, x_n])$, it can be represented in the same way working over $\mathcal{K}_i$, and the two representations must be the same. Hence, all coefficients of f and of g must be in all of the $\mathcal{K}_i$, i.e., in $\mathcal{K}$ which shows that $f/g \in \operatorname{frac}(\mathcal{K}[x_1, \dots, x_n])$, as required.

We shall have to work a great deal harder to prove part (b).