Math 711: Lecture of October 19, 2007

Our next goal is to prove the two results stated at the end of the Lecture Notes of October 17.

Proof of the theorem on preserving the field property for a finite purely inseparable extension. Recall that $\mathcal{L}'$ is a finite purely inseparable extension of $\mathcal{L}$, and $\{\mathcal{L}_i\}_i$ is a family of fields directed by $\supseteq$ whose intersection if $\mathcal{L}$. Fix $\mathcal{L}_0$ in the family: we need only consider fields in the family contained in $\mathcal{L}_0$. Let $\overline{\mathcal{L}_0}$ be an algebraic closure of $\mathcal{L}_0$. Since $\mathcal{L}'$ is purely inseparable over $\mathcal{L}$, $\mathcal{L}'$ may be viewed, in a unique way, as a subfield of $\overline{\mathcal{L}_0}$. Choose a basis $b_1, \ldots, b_h$ for $\mathcal{L}'$ over $\mathcal{L}$. For every i we have a map

$$\mathcal{L}_i \otimes_{\mathcal{L}} \mathcal{L}' \to \mathcal{L}_i[\mathcal{L}'],$$

where the right hand side is the smallest subfield of $\overline{\mathcal{L}_0}$ containing $\mathcal{L}_i$ and $\mathcal{L}'$. The image of this map is evidently a field. Therefore, to prove the Theorem, we need only prove that the map is an isomorphism whenever i is sufficiently small.

Note that the elements $1 \otimes b_j$ span the left hand side as a vector space over $\mathcal{L}_i$. Hence, for every i, the left hand side is a vector space of dimension h over $\mathcal{L}_i$. The image of the map is a ring containing $\mathcal{L}_i$ and the b_j. It therefore contains $\mathcal{L}b_1 + \cdots + \mathcal{L}b_n = \mathcal{L}'$. It follows that the image of the map is $\mathcal{L}_i[\mathcal{L}']$, i.e., the map is onto. The image of the map is spanned by $b_1, \ldots, b_h$ as an $\mathcal{L}_i$-vector space. Therefore, the map is an isomorphism whenever $b_1, \ldots, b_h$ are linearly indpendent over $\mathcal{L}_i$. Choose i so as to make the dimension of the vector space span of $b_1, \ldots, b_h$ over $\mathcal{L}_i$ as large as possible. Since this dimension must be an integer in $\{0, \ldots, h\}$, this is possible. Note that if a subset of the $b_1, \ldots, b_h$ has no nonzero linear relation over $\mathcal{L}_i$, this remains true for all smaller fields in the family.

Call the maximum possible dimension d. By renumbering, if necessary, we may assume that $b_1, \ldots, b_d$ are linearly independent over $\mathcal{L}_i$. We can conclude the proof of the Theorem by showing that $d = h$. If not, b_{d+1} is linearly dependent on $b_1, \ldots, b_d$, so that there is a unique linear relation

$$(*) \quad b_{d+1} = c_1 b_1 + \cdots + c_d b_d,$$

where every $c_j \in \mathcal{L}_i$. Since $b_1, \ldots, b_h$ are linearly independent over $\mathcal{L}$, at least one $c_{j_0} \notin \mathcal{L}$. Choose $\mathcal{L}_{i'} \subseteq \mathcal{L}_i$ such that $c_{j_0} \notin \mathcal{L}_{i'}$. Then $b_1, \ldots, b_{d+1}$ are linearly independent over $\mathcal{L}_{i'}$: if there were a relation different from $(*)$, it would imply the dependence of $b_1, \ldots, b_d$. This contradictis that d is maximum. $\square$

Our next objective is to prove part (b) of the Theorem stated at the top of p. 7 of the Lecture Notes from October 17. We first introduce some terminology. An module C over B (which in the applications here will be a B-algebra) is called *injectively free* over B if for every $u \neq 0$ in C there is an element $f \in \mathrm{Hom}_B(C, B)$ such that $f(u) \neq 0$. This is equivalent to the assumption that C can be embedded in a (possibly infinite) product of copies of B: if ${}_fB = B$ for every $f \in \mathrm{Hom}_B(C, B)$, then

$$C \to \prod_{f \in \mathrm{Hom}_B(C,\, B)} {}_fB$$

is an injection if and only if C is injectively free over B. It is also quite easy to see that C is injectively free over B if and only if the natural map

$$C \to \mathrm{Hom}_B\big(\mathrm{Hom}(C,\, B),\, B\big)$$

from C to its double dual over B is injective.

Note that if C is injectively free over B then $C[x_1, \ldots, x_n]$ is injectively free over $B[x_1, \ldots, x_n]$, and that $C[[x_1, \ldots, x_n]]$ in injectively free over $B[[x_1, \ldots, x_n]]$: choose a nonzero coefficient of u, choose a map $C \to B$ which is nonzero on that coefficient, and then extend it by letting it act on coefficients.

We shall use the notation $\mathcal{F}((x_1, \ldots, x_n))$ for $\mathrm{frac}\,(\mathcal{F}[[x_1, \ldots, x_n]])$ when $\mathcal{F}$ is a field. Note that in case there is just one indeterminates $\mathcal{F}((x)) = \mathcal{F}[[x]][x^{-1}]$ is the ring of Laurent power series in x with coefficients in the field $\mathcal{F}$: any given series contains at most finitely many terms in which the exponent on x is negative, but the largest negative exponent depends on the series under consideration.

We next observe the following fact:

Lemma. *If $B \subseteq C$ are domains, $\mathcal{F} = \mathrm{frac}\,(B)$, C is injectively free over B, and x is a formal indeterminate over C, then*

$$\big(\mathrm{frac}\,(C[[x]])\big) \cap \mathcal{F}((x)) = \mathrm{frac}\,(B[[x]]).$$

Proof. It suffices to show $\subseteq$: the other inclusion is obvious. Suppose that

$$u \in \mathrm{frac}\,(C[[x]]) \cap \mathcal{F}((x)) - \{0\}.$$

Then we can write

$$u = x^h(\sum_{j=0}^{\infty} \beta_j x^j)$$

where $h \in \mathbb{Z}$, the $\beta_j \in \mathcal{F}$, and $\beta_0 \neq 0$. All three fields contain the powers of x, and so we may multiply by x^{-h} without affecting the issue. Thus, we may assume that $h = 0$. We want to show that $u \in \mathrm{frac}\,(B[[x]])$. Since $u \in \mathrm{frac}\,(C[[x]])$, there exists $v \neq 0$ and w in $C[[x]]$ such that $w = vu \in C[[x]]$. Let $v = \sum_{j=0}^{\infty} c_j x^j$ and $w = \sum_{k=0}^{\infty} c'_k x^k$, where the c_j, $c'_j \in C$. Then for each $m \geq 0$, we have that

$$(*) \qquad \sum_{j+k=m} c_j \beta_k = c'_m.$$

Choose j_0 such that $c_{j_0} \neq 0$ and choose $f : C \to B$, B-linear, such that $f(c_{j_0}) \neq 0$ in B. Extend f to a map $C[[x]]$ to $B[[x]]$ by letting it act on coefficients. Then we may multiply the equation $(*)$ by b to get

$$\sum_{j+k=m} c_j(b\beta_k) = bc'_m,$$

and now the B-linearity of f implies that

$$\sum_{j+k=m} f(c_j)b\beta_k = bf(c'_m).$$

Now we may use the fact that b is not a zerodivisor in B to conclude that

$$\sum_{j+k=m} f(c_j)\beta_k = f(c'_m),$$

as we wanted to show. These equations show that $f(v)u = f(w)$, and $f(v) \neq 0$ because $f(c_{j_0}) \neq 0$. Since $f(v)$, $f(w) \in B[[x]]$, we have that $u = f(w)/f(v) \in \operatorname{frac}(B[[x]])$, as required. $\square$

We can now prove part (b) of the Theorem on p. 7 of the Lecture Notes from October 17.

Proof of the theorem on intersections of fraction fields of formal power series rings. We prove the theorem by induction on n. If $n = 1$ it follows from the uniqueness of coefficients in the Laurent expansion of an element of

$$\operatorname{frac}(\mathcal{K}_j([[x]]) = \mathcal{K}_j[[x]][x^{-1}].$$

Now assume the result for $n-1$ variables. For every j, we have

$$\mathcal{K}_j((x_1, \ldots, x_n)) \subseteq \mathcal{K}_j((x_1, \ldots, x_{n-1}))((x_n)).$$

It follows from the one variable case that

$$\bigcap_j \mathcal{K}_j((x_1, \ldots, x_n)) \subseteq \big(\bigcap_j \mathcal{K}_j((x_1, \ldots, x_{n-1}))\big)((x_n)),$$

and from the induction hypothesis that

$$\bigcap_j \mathcal{K}_j((x_1, \ldots, x_{n-1})) = \mathcal{K}((x_1, \ldots, x_{n-1})).$$

Hence,

$$\bigcap_j \mathcal{K}_j((x_1, \ldots, x_n)) \subseteq \mathcal{K}((x_1, \ldots, x_{n-1}))((x_n)).$$

Fix any element j_0 in the index set. Then we have

$$(*) \quad \bigcap_j \mathcal{K}_j((x_1, \ldots, x_n)) \subseteq \mathcal{K}_{j_0}((x_1, \ldots, x_n)) \cap \mathcal{K}((x_1, \ldots, x_{n-1}))((x_n)).$$

We now want to apply the Lemma from the preceding page. Let $B = \mathcal{K}[[x_1, \ldots, x_{n-1}]]$ and $C = \mathcal{K}_{j_0}[[x_1, \ldots, x_{n-1}]]$. Since $\mathcal{K}_{j_0}$ is $\mathcal{K}$-free, it embeds in a direct sum of copies of $\mathcal{K}$ and, hence, in a product of copies of $\mathcal{K}$. Thus, $\mathcal{K}_{j_0}$ is injectively free over $\mathcal{K}$, and it follows that C is injectively free over B. The Lemma from p. 2 applied with $x = x_n$ then asserts precisely that

$$(**) \quad \mathcal{K}_{j_0}((x_1, \ldots, x_n)) \cap \mathcal{K}((x_1, \ldots, x_{n-1}))((x_n)) =$$

$$\operatorname{frac}(C[[x]]) \cap \big(\operatorname{frac}(B)\big)((x_n)) = \operatorname{frac}(B[[x_n]]) = \mathcal{K}((x_1, \ldots, x_n)).$$

From $(*)$ and $(**)$, we have that

$$\bigcap_i \mathcal{K}_i((x_1, \ldots, x_n)) \subseteq \mathcal{K}((x_1, \ldots, x_n)).$$

The opposite inclusion is obvious. $\square$

Corollary. *Let K be a field of characteristic $p > 0$ and let Λ be a p-base for K. Let A be the formal power series ring $K[[x_1, \ldots, x_n]]$. Then*

$$\bigcap_{\Gamma \text{ cofinite in } \Lambda} \operatorname{frac}(A^\Gamma) = \operatorname{frac}(A).$$

Proof. Since the completion of A^Γ is $K^\Gamma[[x_1, \ldots, x_n]]$, we have that

$$(*) \quad \bigcap_{\Gamma \text{ cofinite in } \Lambda} \operatorname{frac}(A^\Gamma) \subseteq \bigcap_{\Gamma \text{ cofinite in } \Lambda} \operatorname{frac}(K^\Gamma[[x_1, \ldots, x_n]]).$$

Since

$$\bigcap_{\Gamma \text{ cofinite in } \Lambda} K^\Gamma = K$$

by the Lemma on p. 6 of the Lecture Notes from October 17 and part (b) of the Theorem on p. 7 of the Lecture Notes from October 17 we have that the right hand term in $(*)$ is $\operatorname{frac}(K[[x_1, \ldots, x_n]])$. This proves one of the inclusions needed, while the opposite inclusion is obvious. $\square$

We also want to observe the following:

Lemma. *Let $\mathcal{L}$ be any field of characteristic $p > 0$, and let Ω be any field finitely generated over $\mathcal{L}$. Then there exists a field $\Omega' \supseteq \Omega$ finitely generated over $\mathcal{L}$ such that Ω' is a finite separable algebraic extension of a pure transcendental extension $\mathcal{L}'(y_1, \ldots, y_h)$ of a field $\mathcal{L}'$ that is a finite purely inseparable algebraic extension of $\mathcal{L}$.*

Proof. Let h be the transcendence degree of Ω over $\mathcal{L}$. Then Ω is a finite algebraic extension of a pure transcendental extension $\mathcal{F} = \mathcal{K}(z_1, \ldots, z_h)$, where $z_1, \ldots, z_h$ is a transcendence basis for Ω over $\mathcal{L}$. Suppose that $\Omega = \mathcal{F}[\theta_1, \ldots, \theta_s]$ where every θ_j is algebraic over $\mathcal{F}$. Within the algebraic closure $\overline{\Omega}$ of Ω, we may form $\mathcal{F}^\infty[\theta_1, \ldots, \theta_s]$, where $\mathcal{F}^\infty$ is the perfect closure of $\mathcal{F}$ in $\overline{\Omega}$. Since $\mathcal{F}^\infty$ is perfect, every θ_i is separable over $\mathcal{F}^\infty$, and so every θ_i satsfies a separable equation over $\mathcal{F}^\infty$. Let $\alpha_1, \ldots, \alpha_N$ be all the coefficients of these equations. Then every θ_i is separable over $\mathcal{F}[\alpha_1, \ldots, \alpha_N]$, and every α_j has a q_j th power in $\mathcal{F}$. Hence, we can choose a single $q = p^e$ such that $\alpha_j^q \in \mathcal{F} = \mathcal{L}(z_1, \ldots, z_h)$ for every j. Every α_j^q can be written in the form

$$\frac{f_j(z_1, \ldots, z_h)}{g_j(z_1, \ldots, z_h)}$$

where $f_j,\, g_j \in \mathcal{L}[z_1, \ldots, z_h]$ and $g_j \neq 0$. Hence, α_j can be written as a rational function in the elements $z_1^{1/q}, \ldots, z_h^{1/q}$ in which the coefficients are the q th roots of the coefficients occurring in f_j and g_j. Let $\mathcal{L}'$ be the field obtained by adjoining all the q th roots of all coefficients of all of the f_j and g_j to $\mathcal{L}$. Let $y_j = z_j^{1/q}$, $1 \leq j \leq h$. Then all of the α_j are in $\mathcal{L}'(y_1, \ldots, y_h)$, and every θ_i satsifies a separable equation over $\mathcal{L}'(y_1, \ldots, y_h)$. But then we may take

$$\Omega' = \mathcal{L}'(y_1, \ldots, y_h)[\theta_1, \ldots, \theta_s],$$

which evdiently contains Ω. $\square$

We are now read to prove the Theorem stated at the top of p. 6 of the Lecture Notes from October 17.

Proof that $\mathrm{frac}\,(A^\Gamma) \otimes_A \Omega$ *is a field for* $\Gamma \ll \Lambda$. We recall that, as usual, K is a field of characteristic $p > 0$, and Λ is p-base for K. Let $\mathcal{L} = \mathrm{frac}\,(A)$, and $\mathcal{L}_\Gamma = \mathrm{frac}\,(A^\Gamma)$. Let Ω be a field finitely generated over $\mathcal{L}$. We want to show that for all $\Gamma \ll \Lambda$, $\mathcal{L}_\Gamma \otimes_{\mathcal{L}} \Omega$ is a field. Since every element of $\mathcal{L}_\Gamma$ has a q th power in $\mathcal{L}$, it is equivalent to show that this ring is reduced: it is purely inseparable over Ω. As in the preceding Lemma, we can choose $\Omega' \supseteq \Omega$ such that Ω' is separable over $\mathcal{L}'(y_1, \ldots, y_h)$, where $\mathcal{L}'$ is a finite purely inseparable extension of $\mathcal{L}$ and $y_1, \ldots, y_h$ are indeterminates over $\mathcal{L}'$. Since $\mathcal{L}_\Gamma$ is flat over the field $\mathcal{L}$, we have that

$$\mathcal{L}_\Gamma \otimes_{\mathcal{L}} \Omega \subseteq \mathcal{L}_\Gamma \otimes_{\mathcal{L}} \Omega',$$

and so it suffices to consider the problem for Ω'.

By the Corollary on p. 4 and the Theorem stated at the bottom of p. 6 of the Lecture Notes from October 17 (which is proved on p. 1 of the notes from this lecture), for all $\Gamma \ll \Lambda$, we have that $\mathcal{L}_\Gamma \otimes_{\mathcal{L}} \mathcal{L}'$ is a field. The ring $\mathcal{G} = \mathcal{L}_\Gamma \otimes_{\mathcal{L}} \mathcal{L}'(y_1, \ldots, y_h)$ is a localization of the polynomial ring $(L_\Gamma \otimes_{\mathcal{L}} \mathcal{L}')[y_1, \ldots, y_h]$. Hence, it is a domain, and therefore a field. Let $\mathcal{F} = \mathcal{L}'(y_1, \ldots, y_n)$. Then Ω' is a finite separable algebraic extension of $\mathcal{F}$, and it suffices to show that $\mathcal{G} \otimes_{\mathcal{F}} \Omega'$ is reduced. This follows from the second Corollary on *p.* 4 of the Lecture Notes of September 19, but we give a separate elementary argument. We can replace $\mathcal{G}$ by its algebraic closure: assume it is algebraically closed. By the theorem on the primitive element, $\Omega' \cong \mathcal{F}[X]/\big(h(X)\big)$, where h is a separable polynomial. Then

$$\mathcal{G} \otimes_{\mathcal{F}} \Omega' \cong \mathcal{G}[X]/\big(h(X)\big),$$

and since h is a separable polynomial, this ring is reduced. $\square$

We have now completed the proof of the Theorem on p. 4 of the Lecture Notes from October 17 concerning properties we can preserve with the gamma construction for Γ sufficiently small but cofinite in Λ.