Math 711: Lecture of October 22, 2007

By the *singular locus* $\mathrm{Sing}\,(R)$ in a Noetherian ring R we mean the set

$$\{P \in \mathrm{Spec}\,(R) : R_P \text{ is not regular}\}.$$

We know that if R is excellent, then $\mathrm{Sing}\,(R)$ is a Zariski closed set, i.e., it has the form $\mathcal{V}(I)$ for some ideal I of R. We say that I *defines* the singular locus in R. Such an ideal I is not unique, but its radical is unique. It follows easily that $c \in \mathrm{Rad}\,(I)$ if and only if R_c is regular.

We next want to prove:

Theorem. *Let K be a field of characteristic p with p-base Λ. Let B be a complete local ring with with coefficient field K. Let R be a ring essentially of finite type over B, and for Γ cofinite in Λ let $R^\Gamma = B^\Gamma \otimes_B R$.*

(a) *If R is regular, then R^Γ is regular for all $\Gamma \ll \Lambda$.*

(b) *If $c \in R$ is such that R_c is regular, then $(R_c)^\Gamma \cong (R^\Gamma)_c$ is regular for all $\Gamma \ll \Lambda$.*

(c) *If I defines the singular locus of R, then for all $\Gamma \ll \Lambda$, IR^Γ defines the singular locus in R^Γ.*

Proof. For every Γ cofinite in Λ, $R \to R^\Gamma$ is purely inseparable, and so we have a homeomorphism $\mathrm{Spec}\,(R^\Gamma)\mathrm{Spec}\,(R) = X$, given by contraction of primes. The unique prime ideal of R^Γ lying over P in R is $\mathrm{Rad}\,(PR^\Gamma)$. See the Proposition on p. 2 of the Lecture Notes from October 1. We identify the spectrum of every R^Γ with X. Let Z_Γ denote the singular locus in R^Γ, and Z the singular locus in R. Since all of these rings are excellent, every singular locus is closed in the Zariski topology. If $R \to S$ is faithfully flat and S is regular then R is regular, by the Theorem on p. 2 of the Lecture Notes of September 19. Thus, a prime Q such that S_Q is regular lies over a prime P in R such that R_P is regular. For $\Gamma \subseteq \Gamma'$ we have maps $R \to R^\Gamma \to R^{\Gamma'}$: both maps are faithfully flat. It follows that $Z \subseteq Z_\Gamma \subseteq Z_{\Gamma'}$ for all $\Gamma \subseteq \Gamma'$.

The closed sets in X have DCC, since ideals of R have ACC. It follows that we can choose Γ cofinite in Λ such that Z_Γ is minimal. Since the sets cofinite in Λ are directed under $\supseteq$, it follows that Z_Γ is minimum, not just minimal. We have $Z \subseteq Z_\Gamma$. We want to prove that they are equal. If not, we can choose Q prime in R^Γ lying over P in R such that R^Γ_Q is not regular but R_P is regular. By part (a) of the Theorem at the bottom of p. 4 of the Lecture Notes from October 17, we can choose $\Gamma_0 \subseteq \Gamma$ cofinite in Λ such that PR^{Γ_0} is prime. This prime will be the contraction Q_0 of Q to R^{Γ_0}. Let R_P have Krull dimension d. In R_P, P has d generators. Hence, $Q_0R_P^{\Gamma_0} = PR_P^{\Gamma_0}$ also has d generators, and it follows that Q_0 itself has d generators. Consequently, we have that $R^{\Gamma_0}_{Q_0}$ is regular, and this means that Z_{Γ_0} is *strictly* smaller than Z_Γ: the point corresponding to P is not in Z_{Γ_0}. This contradiction shows that for all $\Gamma \ll \Lambda$, $Z_\Gamma = Z$. It is immediate that for such a choice of Γ, a prime Q of R^Γ is such that R^Γ_Q is not regular if and only if $R_{Q \cap R}$ is not regular. But this holds if and only if $Q \cap R$ contains I, i.e., if and only if $Q \supseteq I$, which is equivalent to $Q \supseteq IR^\Gamma$. This proves (c).

Part (a) is simply the case where Z is empty. Note that for any $c \in R$,

$$(R_c)^\Gamma = B^\Gamma \otimes_B R_c \cong B^\Gamma \otimes_B (R \otimes_R R_c) \cong (B^\Gamma \otimes_B R) \otimes_R R_c \cong (R^\Gamma)_c.$$

Thus, (b) follows from (a) applied to R_c. $\square$

We are now in a position to fill in the details of the proof of the Theorem on the existence of completely stable big test elements stated on p. 2 of the Lecture Notes from October 12. The proof was sketched earlier to motivate our development of the gamma construction.

We need one small preliminary result.

Lemma. *If R is essentially of finite type over B and $B \to C$ is geometrically regular, then $C \otimes_B R$ is geometrically regular over R.*

Proof. This is a base change, so the map is evidently flat. Let P be a prime ideal of R lying over $\mathfrak{p}$ in B. Then

$$\kappa_P \otimes_R (R \otimes_B C) \cong \kappa_P \otimes_B C \cong \kappa_P \otimes_{\kappa_\mathfrak{p}} (\kappa_\mathfrak{p} \otimes_B C).$$

Let $T = \kappa_\mathfrak{p} \otimes_B C$, which is a geometrically regular $\kappa_\mathfrak{p}$-algebra by the hypothesis on the fibers. Then all we need is that every finite algebraic purely inseparable extension field extension $\mathcal{L}$ of κ_P, the ring $\mathcal{L} \otimes_{\kappa_\mathfrak{p}} T$ is regular. We may replace $\mathcal{L}$ by a larger field finitely generated over $\kappa_\mathfrak{p}$. By the Lemma at the top of p. 5 of the Lecture Notes of October 19, we may assume this larger field is a finite separable algebraic extension of $\mathcal{K}(y_1, \ldots, y_h)$, where $\mathcal{K}$ is a finite algebraic purely inseparable extension of $\kappa_\mathfrak{p}$ and $y_1, \ldots, y_h$ are indeterminates. Then $\mathcal{K} \otimes_{\kappa_\mathfrak{p}} T$ is regular by the hypothesis of geometric regularity of the fiber T over $\kappa_\mathfrak{p}$. Therefore, $\mathcal{K}(y_1, \ldots, y_h) \otimes_{\kappa_\mathfrak{p}} T$ is regular because it is a localization of the polynomial ring $(\mathcal{K} \otimes_{\kappa_\mathfrak{p}} T)[y_1, \ldots, y_h]$. Since $\mathcal{L}$ is finite separable algebraic over $\mathcal{K}(y_1, \ldots, y_h)$, the result now follows from the second Corollary on p. 4 of the Lecture Notes from September 19. $\square$

We now restate the result that we want to prove.

Theorem. *Let R be a Noetherian ring of prime characteristic $p > 0$. Suppose that R is reduced and essentially of finite type over an excellent semilocal ring B. Then there are elements $c \in R^\circ$ such that R_c is regular, and every such element c has a power that is a completely stable big test element.*

Proof. By the Lemma above, $\widehat{B} \otimes_B R$ is geometrically regular over R. Moreover, the localization at c may be viewed as has a regular base R_c, and the fibers of $R_c \to \widehat{B} \otimes_R R_c$

are still regular: they are a subset of the original fibers, corresponding to primes of R that do not contain c. By the first Corollary at the top of p. 4 of the Lecture Notes of September 19, $(\widehat{B} \otimes_B R)_c$ is regular. Since R is reduced and $c \in R^\circ$, c is not a zerodivisor in R, i.e., $R \subseteq R_c$. It follows that $\widehat{B} \otimes_B R \subseteq \widehat{B} \otimes_B R_c$, and so $\widehat{B} \otimes_B R$ is reduced. Since $R \to \widehat{B} \otimes_B R$ is faithfully flat, it suffices to prove the result for $\widehat{B} \otimes_B R$, by part (b) of the Proposition at the bottom of p. 8 of the Lecture Notes from September 17.

Thus, we may replace B by its completion. Henceforth, we assume that B is complete. B is now a product of local rings. R is a product in a corresponding way, and every R-module is a product of R-modules over the factors. The hypotheses are preserved on each factor ring, and all of the issues under consideration reduce to consideration of the factors separately. Therefore we need only consider the case where B is a complete local ring.

Choose a coefficient field K for B, and a p-base Λ for K, so that we may use the gamma construction on B. For all $\Gamma \ll \Lambda$, we have that $R^\Gamma = B^\Gamma \otimes_B R$ is reduced, and that

$$B^\Gamma \otimes_B R_c \cong R_c^\Gamma$$

is regular. Since R^Γ is faithfully flat over R, it suffices to consider R^Γ instead of R. Since R^Γ is F-finite, the result is now immediate from the Theorem at the bottom of p. 4 of the Lectture Notes of October 1. $\square$

We want to improve the result above: it will turn out that it suffices to assume that R_c is Gorenstein and weakly F-regular. We will need some further results about weak F-regularity in the Gorenstein case. In particular, we want to prove that when the ring is F-finite, weak F-regularity implies strong F-regularity.

We first note the following fact:

Proposition. *Let R be a Noetherian ring of prime characteristic $p > 0$. Then the following conditions are equivalent:*

(a) *If $N \subseteq M$ are arbitary modules (with no finiteness condition), then N is tightly closed in M.*

(b) *For every maximal ideal m of R, 0 is tightly closed (over R) in $E_R(R/m)$.*

(c) *For every maximal ideal m of R, if u generates the socle in $E(R/m)$, then u is not in the tight closure (over R) of 0 in $E_R(R/m)$.*

Proof. Evidently (a) $\Rightarrow$ (b) $\Rightarrow$ (c). But (c) $\Rightarrow$ (b) is clear, because if if the tight closure of 0 is not 0, it must contain the socle: $R/m \hookrightarrow E_R(R/m)$ is essential, and every nonzero submodule of $E_R(R/m)$ therefore contains u.

Now suppose that $N \subseteq M$ and $u \in M$ is such that $u \in N^*_M - N$. We may replace N by a submodule of M maximal with respect to containing N and not containing u, by Zorn's Lemma. Then we may replace u and $N \subseteq M$ by the image of u in M/N and $0 \subseteq M/N$. Hence, we may assume that $u \in 0^*_M - \{0\}$ and that u is in every nonzero submodule of M. We may now apply the Lemma on p. 1 of the Lecture Notes from September 17 to conclude that for every finitely generated nonzero submodule of M, there is only one associated prime, m, which is maximal, and that the socle is one-dimensional and generated by u. But then the same conclusion applies to M itself, and so M is an essential extension of $Ru \cong Ku$, where $K = R/m$. Hence, M embeds in $E_R(R/m) = E$ so that u generates the socle in E, and $u \in 0^*_M$ implies that $u \in 0^*_E$. $\square$

We next want to prove the following:

Theorem. *Let $(R,\, m,\, K)$ be a Gorenstein local ring of prime characteristic $p > 0$. Then the conditions of the preceding Proposition hold if and only if R is weakly F-regular.*

Moreover, if R is weakly F-regular and F-finite, then R is strongly F-regular.

It will be a while before we can give a complete proof of this result. Our proof of the Theorem requires understanding $E_R(K)$ when R is a Gorenstein local ring.

Calculation of the injective hull of a Gorenstein local ring

Theorem. *Let $(R,\, m,\, K)$ be a Gorenstein local ring with system of parameters $x_1,\, \dots,\, x_n$. For every integer $t \geq 1$, let $I_t = (x_1^t,\, \dots,\, x_n^t)R$. Let $y = x_1 \cdots x_n$. Then*

$$E_R(K) \cong \varinjlim_t R/I_t,$$

where the map $R/I_t \to R/I_{t+1}$ is induced by multiplication by y on the numerators.

Moreover, if $u \in R$ represents a socle generator in $R/(x_1,\, \dots,\, x_n)R$, then for every t, $y^{t-1}u \in R/I_t$ represents the socle generator in R/I_t and in $E_R(K)$.

Proof. Let $E = E_K(R)$ be a choice of injective hull for K. Then $E_t = \mathrm{Ann}_E I_t$ is an injective hull for K over R/I_t, and so is isomorphic to R/I_t. Since every element of E is killed by a power of m, each element of E is some E_t. Then

$$E = \bigcup_t E_t$$

shows that there is some choice of injective maps

$$\theta_t : R/I_t \to R/I_{t+1}$$

such that

$$E = \varinjlim_t E_t,$$

using the maps θ_t. One injection of R/I_t into R/I_{t+1} is given by the map η_t induced by multiplication by y on the numerators: see the Theorem at the bottom of p. 5 of the

Lecture Notes from October 8, applied to $x_1, \dots, x_n$ and $x_1^t, \dots, x_n^t$, with the matrix $A = \mathrm{diag}(x_1^{t-1}, \dots, x_n^{t-1})$. (See also the last statement of the Proposition near the top of p. 8 in the same lecture, which will prove the final statement of the Theorem.) Since the modules have finite lengths, an injection of E_t into E_{t+1} must have image $E_t = \mathrm{Ann}_E I_t$, since the image is clear contained in E_t, and so there must be an automorphism α_t of E_t such that $\theta_t = \alpha_t \circ \eta_t$. In fact, $\alpha_t \in \mathrm{Hom}_{R/I_t}(E_t, E_t) \cong R_t$ must be multiplication by a unit of R_t. Thus, every α_t lifts to a unit $a_t \in R$. Let $b_1 = 1$, and let $b_t = a_1 \cdots a_{t-1}$.

We can now construct a commutative diagram

$$\begin{array}{ccccccccccc}
E_1 & \xrightarrow{\eta_1} & E_2 & \xrightarrow{\eta_2} & \cdots & \xrightarrow{\eta_{t-1}} & E_t & \xrightarrow{\eta_t} & E_{t+1} & \xrightarrow{\eta_{t+1}} & \cdots \\
{\scriptstyle b_1}\big\downarrow & & {\scriptstyle b_2}\big\downarrow & & & & {\scriptstyle b_t}\big\downarrow & & {\scriptstyle b_{t+1}}\big\downarrow & & \\
E_1 & \xrightarrow{\theta_1} & E_2 & \xrightarrow{\theta_2} & \cdots & \xrightarrow{\theta_{t-1}} & E_t & \xrightarrow{\theta_t} & E_{t+1} & \xrightarrow{\theta_{t+1}} & \cdots
\end{array}$$

Commutativity follows from the fact that on E_t, $b_{t+1}\eta_t$ is induced by multiplication by $b_{t+1}y = a_t b_t y = (a_t y)b_t$, and θ_t is induced by multiplication by $a_t y$ on E_t. Since the vertical arrows are isomorphisms, the direct limits are isomorphic. The direct limit of the top row is the module that we are trying to show is isomorphic to E, while the direct limit of the bottom row is E. $\square$