Math 711: Lecture of October 24, 2007

The action of Frobenius on the injective hull of the residue class field of a Gorenstein local ring

Let $(R,\, m,\, K)$ be a Gorenstein local ring of prime characteristic $p > 0$, and let $x_1,\, \ldots\, , x_n$ be a system of parameters. Let $I_t = (x_1^t,\, \ldots\, , x_n^t)R$ for all $t \geq 1$, and let $u \in R$ represent a socle generator in R/I, where $I = I_1 = (x_1,\, \ldots\, , x_n)R$. Let $y = x_1 \,\cdots\, x_n$. We have seen that

$$E = \varinjlim_t R/I_t$$

is an injective hull of $K = R/m$ over R, where the map $R/I_t \to R/I_{t+1}$ is induced by multiplication by y acting on the numerators. Each of these maps is injective. Note that the map from $R/I_t \to R/I_{t+k}$ in the direct limit system is induced by multiplication by y^k acting on the numerators.

Let $e \in \mathbb{N}$ be given. We want to understand the module $\mathcal{F}^e(E)$, and we also want to understand the qth power map $v \mapsto v^q$ from E to $\mathcal{F}^e(E)$. If $r \in R$, we shall write $\langle r;\, x_1^t,\, \ldots\, , x_n^t\rangle$ for the image of r under the composite map $R \twoheadrightarrow R/I_t \hookrightarrow E$, where the first map is the quotient surjection and the second map comes from our construction of E as the direct limit of the R/I_t. With this notation,

$$\langle r;\, x_1^t,\, \ldots\, , x_n^t\rangle = \langle y^k r;\, x_1^{t+k},\, \ldots\, , x_n^{t+k}\rangle$$

for every $k \in \mathbb{N}$.

Since tensor products commute with direct limit, we have that

$$\mathcal{F}^e(E) = \varinjlim_t \mathcal{F}^e(R/I_t) = \varinjlim_t R/(I_t)^{[q]} = \varinjlim_t R/I_{tq}.$$

In the rightmost term, the map from $R/I_{tq} \to R/I_{(t+1)q} = I_{tq+q}$ is induced by multiplication by t^q acting on the numerators. The rightmost direct limit system consists of of a subset of the terms in the system $\varinjlim_t R/I_t$, and the maps are the same. The indices that occur are cofinal in the positive integers, and so we may idenitfy $\mathcal{F}^e(E)$ with E. Under this identification, if $v = \langle r;\, x_1^t,\, \ldots\, , x_n^t\rangle$, then $v^q = \langle r^q;\, x_1^{qt},\, \ldots\, , x_n^{qt}\rangle$.

We can now prove the assertions in the first paragraph of the Theorem on p. 4 of the Lecture Notes from October 22.

Proof that 0 *is tightly closed in* $E_R(K)$ *for a weakly F-regular Gorenstein local ring.* Let $(R,\, m,\, K)$ be a Gorenstein local ring of prime characteristic $p > 0$. We want to determine when $v = \langle u;\, x_1,\, \ldots\, , x_n\rangle$ is in 0^* in E. This happens precisely when there is an element $c \in R^\circ$ such that $cv^q = 0$ in $\mathcal{F}^e(E)$ for all $q \gg 0$. But $cv^q = \langle cu^q;\, x_1^q,\, \ldots\, , x_n^q\rangle$, which is 0 if and only if $cu^q \in I_q = I^{[q]}$ for all $q \gg 0$. Thus, 0 is tightly closed in E if and only if I is tightly closed in R. This gives a new proof of the result that in a Gorenstein local ring, if I is tightly closed then R is weakly F-regular. But it also proves that if I is tightly closed, every submodule of every module is tightly closed. In particular, if R is weakly F-regular then every submodule over every module is tightly closed. $\square$

It remains to show that when a Gorenstein local ring is F-finite, it is strongly F-regular. We first want to discuss some issues related to splitting a copy of local ring from a module to which it maps.

Splitting criteria and approximately Gorestein local rings

Many of the results of this section do not depend on the characteristic.

Theorem. *Let (R, m, K) be a local ring and M an R-module. Let $f : R \to M$ be an R-linear map. Suppose that R is complete or that M is finitely generated. Let E denote an injective hull for the residue class field $K = R/m$ of R. Then $R \to M$ splits if and only if the map $E = E \otimes_R R \to E \otimes_R M$ is injective.*

Proof. Evidently, if the map splits the map obtained after tensoring with E (or any other module) is injective: it is still split. This direction does not need any hypothesis on R or M. For the converse, first consdider the case where R is complete. Since the map $E \otimes_R R \to E \otimes_R M$ is injective, if we apply $\mathrm{Hom}_R(\_\,, E)$, we get a surjective map. We switch the order of the modules in each tensor product, and have that

$$\mathrm{Hom}_R(R \otimes_E E,\, E) \to \mathrm{Hom}_R(M \otimes_R E,\, E)$$

is surjective. By the adjointness of tensor and Hom, this is isomorphic to the map

$$\mathrm{Hom}_R\big(M,\, \mathrm{Hom}_R(E,\, E)\big) \to \mathrm{Hom}_R(R,\, \mathrm{Hom}_R(E,\, E).$$

By Matlis duality, we have that $\mathrm{Hom}_R(E, E)$ may be naturally identified with R, since R is complete, and this yields that the map $\mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R(R, R)$ induced by composition with $f : R \to M$ is surjective. An R-linear homomorphism $g : M \to R$ that maps to the identity in $\mathrm{Hom}_R(R, R)$ is a splitting for f.

Now supppose that R is not necessarily complete, but that M is finitely generated. By part (b) of the Theorem on p. 3 of the Lecture Notes from Septmeber 24, completing does not affect whether the map splits. The result now follows from the complete case, because E is the same for R and for $\widehat{R}$, and $E \otimes_{\widehat{R}} (\widehat{R} \otimes_R \_\,)$ is the same as $E \otimes_R \_$ by the associativity of tensor. $\square$

This result takes a particularly concrete form in the Gorenstein case.

Theorem (splitting criterion for Gorenstein rings). *Let $(R,\, m,\, K)$ be a Gorenstein local ring, and let $x_1, \ldots, x_n$ be a system of parameters for R. Let $u \in R$ represent a socle generator in R/I, where $I = (x_1, \ldots, x_n)$, let $y = x_1 \cdots, y_n$, and let $I_t = (x_1^t, \ldots, x_n^t)R$ for $I \geq 1$. Let $f : R \to M$ be an R-linear map with $f(1) = w \in M$, and assume either that R is complete or that M is finitely generated. Then the following conditions are equivalent:*

(1) *$f : R \to M$ is split.*

(2) *For every ideal J of R, $R/J \to M/JM$ is injective, where the map is induced by applying $(R/J) \otimes_R \_$.*

(3) *For all $t \geq 1$, $R/I_t \to M/I_tM$ is injective.*

(4) *For all $t \geq 1$, $y^{t-1}uw \notin I_tM$.*

Moreover, if $x_1, \ldots, x_n$ is a regular sequence on M, then the following two conditions are also equivalent:

(5) *$R/I \to R/IM$ is injective.*

(6) *$uw \notin IM$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) is clear. The map $R/I_t \to M/I_tM$ has a nonzero kernel if and only if the socle element, which is the image of $y^{t-1}u$, is killed, and this element maps to $y^{t-1}uw$. Thus, the statements in (3) and (4) are equivalent for every value of t, and the equivalence (5) $\Leftrightarrow$ (6) is the case $t = 1$. We know from the preceding Theorem that $R \to M$ is split if and only if $E \to E \otimes_R M$ is injective, and this map is the direct limit of the maps $R/I_t \to (R/I_t) \otimes_R M$ by the Theorem on p. 2. This shows that (3) $\Rightarrow$ (1). Thus, (1), (2), (3), and (4) are all equivalent and imply (5) and (6), while (5) and (6) are also equivalent. To complete the proof it suffices to show that (6) $\Rightarrow$ (4) when $x_1, \ldots, x_n$ is a regular sequence on M. Suppose $y^tuw \in (x_1^t, \ldots, x_n^t)M$. Then $uw \in (x_1^t, \ldots, x_n^t)M :_M y^t = (x_1, \ldots, x_n)M$ by the Theorem on p. 3 of the Lecture Notes from October 8. $\square$

Remark. If $M = S$ is an R-algebra and the map $R \to S$ is the structural homomorphism, then the condition in part (2) is that every ideal J of R is contracted from S. Similarly, the condition in (4) (respectively, (5)) is that I_t (respectively, I) be contracted from S.

We define a local ring $(R,\, m,\, K)$ to be *approximately Gorenstein* if there exists a decreasing sequence of m-primary ideals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_t \supseteq \cdots$ such that every R/I_t is a Gorenstein ring (i.e., the socle of every R/I_t is a one-dimensional K-vector space) and the I_t are cofinal with the powers of m. That is, for every $N > 0$, $I_t \subseteq m^N$ for all $t \gg 1$. Evidently, a Gorenstein local ring is approximately Gorenstein, since we may take $I_t = (x_1^t, \ldots, x_n^t)R$, where $x_1, \ldots, x_n$ is a system of parameters.

Note that the following conditions on an m-primary ideal I in a local ring $(R,\, m,\, K)$ are equivalent:

(1) R/I is a 0–dimensional Gorenstrein.

(2) The socle in R/I is one-dimensional as a K-vector space.

(3) I is an irreducible ideal, i.e., I is not the intersection of two strictly larger ideals.

Note that (2) $\Rightarrow$ (3) because when (2) holds, any two larger ideals, considered modul I, must both contain the socle of R/I. Conversely, if the socle of R/I has dimension 2 or more, it contains nonzero vector subspaces V and V' whose intersection is 0. The inverse iamges of V and V' in R are ideals strictly larger than I whose intersection is I. $\square$

If R itself has dimension 0, the chain I_t is eventually 0, and so in this case an approximately Gorenstein ring is Gorenstein. In higher dimension, it turns out to be a relatively weak condition on R.

Theorem. *Let $(R,\, m,\, K)$ be a local ring. Then R is approximately Gorenstein if and only if $\widehat{R}$ is approximately Gorenstein. Moreover, R is approximately Gorenstein provided that at least one of the following conditions holds:*

(1) *$\widehat{R}$ is reduced.*

(2) *R is excellent and reduced.*

(3) *R has depth at least 2.*

(4) *R is normal.*

The fact that the condition holds for R if and only it holds for $\widehat{R}$ is obvious. Moreover, (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3). We shall say more about why the Theorem given is true in the sequel. For a detailed treatment see [M. Hochster, *Cyclic purity versus purity in excellent Noetherian rings*, Trans. Amer. Math. Soc. **231** (1977) 463–488.], which gives the following precise characterization: a local ring of dimension at least one is approximately Gorenstein if and only if R has positive depth and there is no associated prime P of the completion $\widehat{R}$ such that $\dim(\widehat{R}/P) = 1$ and $(\widehat{R}/P) \oplus (\widehat{R}/P)$ embeds in $\widehat{R}$.

Before studying characterizations of the property of being approximately Gorenstein further, we want to note the folliwing.

Propositon. *Let $(R,\, m,\, K)$ be an approximately Gorenstein local ring and let $\{I_t\}_t$ be a descending chain of m-primary irreducible ideals cofinal with the powers of m. Then an injective hull $E = E_R(K)$ is an increasing union $\bigcup_t \mathrm{Ann}_{I_t} E$, and $\mathrm{Ann}_E I_t \cong R/I_t$, so that E is the direct limit of a system in which the modules are the R/I_t and the maps are injective.*

Proof. Since every element of E is killed by a power of m, every element of E is in $\mathrm{Ann}_E I_t$ for some t. We know that $\mathrm{Ann}_E I_t$ is an injective hull for K over R/I_t. Since R/I_t is 0-dimensional Gorenstein, this ring itself is an injective hull over itself for K. $\square$

This yields:

Theorem. *Let $(R,\, m,\, K)$ be an approximately Gorenstein local ring and let $\{I_t\}_t$ be a descending chain of m-primary irreducible ideals cofinal with the powers of m. Let $u_t \in R$*

represent a socle generator in R/I_t. Let $f : R \to M$ be an R-linear map with $f(1) = w \in M$. Then the following conditions are equivalent:

(1) *$f : R \to M$ splits over R.*

(2) *For all $t \geq 1$, $R/I_t \to M/I_tM$ is injective.*

(3) *For all $t \geq 1$, $u_t w \notin I_tM$.*

Proof. Since $E = E_R(K)$ is the direct limit of the R/I_t, we may argue exactly as in the proof of the Theorem at the top of p 3. $\square$

When is a ring approximately Gorenstein?

To prove a sufficient condition for a local ring to be approximately Gorenstein, we want to introduce a corresponding notion for modules. Let (R, m, K) be local and let M be a finitely generated R-module. We shall say that $N \subseteq M$ is *cofinite* if M/N is killed by power of m. (The reader should be aware that the term "cofinite module" is used by some authors for a module with DCC.) The following two conditions on a cofinite submodule are then equivalent, just as in the remark at the bottom of p. 3 and top of p. 4.

(1) The socle in M/N is one-dimensional as K-vector space.

(2) N is in irreducible submodule of M, i.e., it is not the intersection of two strictly larger submodules of M.

We shall say that M has *small cofinite irreducibles* if for every positive integer t there is an irreducible cofinite submodule N of M such that $N \subseteq m^tM$. Thus, a local ring R is approximately Gorenstein if and only if R itself has small cofinite irreducibles.

Note the the question of whether $(R,\, m,\, K)$ is approximately Gorenstein or whether M has small cofinite irreducibles is unaffected by completion: there is a bijection between the cofinite submodules N of M and those of $\widehat{M}$ given by letting N correspond to $\widehat{N}$. The point is that if N' is cofinite in $\widehat{M}$, $\widehat{M}/N'$ is a finitely generated R-module (in fact, it has finite length) and $M \to \widehat{M}/N'$ is surjective, since $M/m^tM \cong \widehat{M}/m^t\widehat{M}$ for all t, so that N' is the completion of $N' \cap M$. Moreover, when N and N' correspond, $M/N \cong \widehat{M}/N'$ since M/N is already a complete R-module. In particular, irreducibility is preserved by the correspondence.

We have already observed that Gorenstein local rings are approximately Gorenstein. We next note:

Proposition. *Let $(R,\, m,\, K)$ be a local ring. If M is a finitely generated R-module that has small cofinite irreducibles, then every nonzero submodule of M has small cofinite irreducibles.*

Proof. Suppose that $N \subseteq M$ is nonzero. By the Artin-Rees lemma there is a constant $c \in \mathbb{N}$ such that $m^t M \cap N \subseteq m^{t-c} N$ for all $t \geq c$. If M_{t+c} is cofinite in M and such that $M_{t+c} \subseteq m^{t+c} M$ and M/M_{t+c} has a one-dimensional socle, then $N_t = M_{t+c} \cap N$ is cofinite in N, contained in $m^t N$ (so that N/N_t is nonzero) and has a one-dimensional socle, since N/N_t embeds into M/M_{t+c}. $\square$

Before giving the main result of this section, we note the following fact, due to Chevalley, that will be needed in the argument.

Theorem (Chevalley's Lemma). *Let M be a finitely generated module over a complete local ring (R, m, K) and let $\{M_t\}_t$ denote a nonincreasing sequence of submodules. Then $\bigcap_t M_t = 0$ if and only if for every integer $N > 0$ there exists t such that $M_t \subseteq m^N M$.*

Proof. The "if" part is clear. Suppose that the intersection is 0. Let $V_{t,N}$ denote the image of M_t in $M/m^N M$. Then the $V_{t,N}$ do not increase as t increases, and so are stable for all large t. Call the stable image V_N. Then the maps $M/m^{N+1}M \to M/m^N M$ induce surjections $V_{N+1} \twoheadrightarrow V_N$. The inverse limit W of the V_N may be identified with a submodule of the inverse limit of the $M/m^N M$, i.e. with a submodule of M, and any element of W is in

$$\bigcap_{t,N} (M_t + m^N M) = \bigcap_t \big(\bigcap_N (M_t + m^N M) \big) = \bigcap_t M_t.$$

If any V_{N_0} is not zero, then since the maps $V_{N+1} \twoheadrightarrow V_N$ are surjective for all N, the inverse limit W of the V_N is not zero. But V_N is zero if and only if $M_t \subseteq m^N M$ for all $t \gg 0$. $\square$

The condition given in the Theorem immediately below for when a finitely generated module of positive dimension over a complete local ring has small cofinite irreducibles is necessary as well as sufficient: we leave the necessity as an exercise for the reader. The proof of the equivalence is given in [M. Hochster, *Cyclic purity versus purity in excellent Noetherian rings*, Trans. Amer. Math. Soc. **231** (1977) 463–488.]

Theorem. *Suppose that M is a finitely generated module over a complete local ring (R, m, K) such that* $\dim M \geq 1$. *Suppose that m is not an associated prime of M and that if P is an associated prime of M such that* $\dim R/P = 1$ *then $R/P \oplus R/P$ is not embeddable in M. Then M has small cofinite irreducibles.*

Proof. We use induction on $\dim M$. First suppose that $\dim M = 1$. We represent the ring R as a homomorphic image of a complete regular local ring S of dimension d. Because R is catenary and $\dim M = 1$, the annihilator of M must have height $d-1$. Choose part of a system of parameters $x_1,\, \ldots\,, x_{d-1}$ in the annihilator. Now view M as a module over $R' = S/(x_1,\, \ldots\,, x_{d-1})$. We change notation and simply write R for this ring. Then

R is a one-dimensional complete local ring, and R is Gorenstein. It follows that R has small cofinite irreducibles, and we can complete the argument, by the Proposition on the preceding page, by showing that M can be embedded in R. Note that for any minimal prime $\mathfrak{p}$ in R, $R_{\mathfrak{p}}$ is a (zero-dimensional) Gorenstein ring. (In fact, any localization of a Gorenstein local ring at a prime is again Gorenstein: but we have not proved this here. However, in this case, we may view $R_{\mathfrak{p}}$ as the quotient of the regular ring $S_{\mathfrak{q}}$, where $\mathfrak{q}$ is the inverse image of $\mathfrak{p}$ in S, by an ideal generated by a system of parameters for $S_{\mathfrak{q}}$, and the result follows.)

To prove that we can embed M in R, it suffices to show that if $W = R^\circ$, then $W^{-1}M$ can be embedded in $W^{-1}R$. One then has $M \subseteq W^{-1}M \subseteq W^{-1}R$, and the values of the injective map $M \hookrightarrow W^{-1}R$ on a finite set of generators of M involve only finitely many elements of W. Hence, one can multiply by a single element of W, and so arrange that $M \hookrightarrow W^{-1}R$ actually has values in R.

But $W^{-1}R$ is a finite product of local rings $R_{\mathfrak{p}}$ as $\mathfrak{p}$ runs through the minimal primes of R, and so it suffices to show that if $\mathfrak{p}$ is a minimal prime of R in the support of M, then $M_{\mathfrak{p}}$ embeds in $R_{\mathfrak{p}}$. Now, $M_{\mathfrak{p}}$ has only $\mathfrak{p}R_{\mathfrak{p}}$ as an associated prime, and since only one copy of $R/\mathfrak{p}$ can be embedded in M, only one copy of $\kappa_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ can be embedded in $M_{\mathfrak{p}}$. Thus, $M_{\mathfrak{p}}$ is an essential extension of a copy of $\kappa_{\mathfrak{p}}$. Thus, it embeds in the injective hull of the residue field of $R_{\mathfrak{p}}$, which, since $R_{\mathfrak{p}}$ is a zero-dimensional Gorenstein ring, is the ring $R_{\mathfrak{p}}$ itself.

Now suppose that $\dim M = d > 1$ and that the result holds for modules of smaller dimension. Choose a maximal family of prime cyclic submodules of M, say $Ru_1, \ldots, Ru_s$, such that $\mathrm{Ann}_R u_i$ is a prime Q_i for every i and the sum $N = Ru_1 \oplus \cdots \oplus Ru_s$ is direct. Then M is an essential extension of N: if $v \in M$, it has a nonzero multiple rv that generates a prime cyclic module, and if this prime cyclic module does not meet N we can enlarge the family. Since M is an essential extension of N, M embeds in the injective hull of N, which we may identify with the direct sum of the $E_i = E_R(Ru_i)$. Note that a prime ideal of R may occur more than once among the Q_i, but not if $\dim(R/Q_i) = 1$, and R/m does not occur. Take a finite set of generators of M. The image of each generator only involves finitely many elements from a given E_i. Let M_i be the submodule of E_i generated by these elements. Then $M_i \subseteq E_i$, so that $\mathrm{Ass}\,(M_i) = Q_i$, and M_i is an essential extension of R/Q_i. What is more $M \subseteq \oplus_{i=1}^s M_i$.

By the Propositiion at the bottom of p. 5, it suffices to show that this direct sum, which satisfies the same hypotheses as M, has small cofinite irreducibles. Thus, by we need only consider the case where $M = \bigoplus_{i=1}^s M_i$ as described. We assume that, for $i \leq h$, $\mathrm{Ass}\, M_i = \{Q_i\}$ with $\dim(R/Q_i) = 1$ and with the Q_i *mutually distinct*, while for $i > h$, $\dim(R/Q_i) > 1$, and these Q_i need not all be distinct. Now choose primes $P_1, \ldots, P_s$ such that, for every i, $\dim R/P_i = 1$, such that $P_1, \ldots, P_s$ are all distinct, and such that for all i, $P_i \supseteq Q_i$. We can do this: for $1 \leq i \leq h$, the choice $P_i = Q_i$ is forced. For $i > h$ we can solve the problem recursively: simply pick P_i to be any prime different from the others alreaday selected and such that $P_i \supseteq Q_i$ and $\dim R/P_i = 1$. (We are using the fact that a local domain R/Q of dimension two or more contains infinitely many primes P such that $\dim R/P = 1$. To see this, kill a prime to obtain a ring of dimension exactly two. We then need to see that there are infinitely many height one primes. But if there are only finitely many, their union cannot be the entire maximal ideal, and a minimal prime of an element of the maximal ideal not in their union will be another height one prime.)

Fix a positive integer t. We shall construct a submodule N of M contained in m^tM and such that M/N is cofinite with a one-dimensional socle. We shall do this by proving that for every i there is a submodule N_i of M_i with the following properties:

(1) $N_i \subseteq m^t M_i$

(2) $\operatorname{Ass} M_i/N_i = \{P_i\}$ and M_i/N_i is an essential extension of R/P_i.

It then follows that $\overline{M} = M/(\bigoplus_i N_i)$ is a one-dimensional module with small cofinite irreducibles, and so we can choose $\overline{N} \subseteq m^t\overline{M}$ such that $\overline{M}/\overline{N}$ has finite length and a one-dimensional socle. We can take N to be the inverse image of $\overline{N}$ in M. This shows that the problem reduces to the construction of the N_i with the two properties listed.

If $i \le h$ we simply take $N_i = 0$. Now suppose that $i > h$. To simplify notation we write M, Q and P for M_i, Q_i and P_i, respectively. Let $D_k \subseteq M$ be the contraction of P^kM_P to $M \subseteq M_P$. Since $\bigcap_k P^kM_P = 0$ (thinking over R_P), we have that $\bigcap_k D_k = 0$. Since M is complete, by Chevalley's Lemma, we can choose k so large that $D_k \subseteq m^tM$.

We shall show that the completion of M_P over the completion of R_P satisfies the hypothesis of the Theorem. But then, since M_P and its completion have dimension strictly smaller than M, it follows from the induction hypothesis that, working over R_P, M_P has small cofinite irreducibles. Consequently, we may choose a cofinite irreducible $N' \subseteq P^kM_P$, and the contraction of N' to M will have all of the properties that we want, since it will be contained in $D_k \subseteq m^tM$.

Thus, we need only show that the completion of M_P over the completion of R_P satisfies the hypothesis of the Theorem. Since $\operatorname{Ass}(M) = \{Q\}$, we have that $\operatorname{Ass}(M_P) = \{QR_P\}$, and so PR_P is not an associated prime of M_P. Thus, the depth of M_P is at least one, and this is preserved when we complete. By Problem 2(b) of Problem Set #3, $\operatorname{Ass}(\widehat{M_P})$ is the same as the set of associated primes of the completion of R_P/QR_P, which we may identify with $\widehat{R_P}/Q\widehat{R_P}$. Since this ring is reduced, the primes $\mathfrak{q}$ that occur are minimal primes of $Q\widehat{R_P}$. For such a prime $\mathfrak{q}$,

$$\operatorname{Ann}_{\widehat{M_P}}\mathfrak{q} \subseteq \operatorname{Ann}_{\widehat{M_P}}Q \cong \widehat{R_P} \otimes_{R_P} \operatorname{Ann}_{M_P}QR_P,$$

since $\widehat{R_P}$ is flat over R_P. From the hypothesis, we know that $\operatorname{Ann}_{M_P}QR_P$ has torsion free rank one over R_P/QR_P, and so it embeds in R_P/QR_P. It follows that $\operatorname{Ann}_{\widehat{M_P}}\mathfrak{q}$ embeds in $\widehat{R_P}/Q\widehat{R_P}$. Since this ring is reduced with $\mathfrak{q}$ as one of the minimal primes, its total quotient ring is a product of fields. Hence, it is not possible to embed the direct sum of two copies of $(\widehat{R_P}/Q\widehat{R_P})/\mathfrak{q}$ in $\widehat{R_P}/Q\widehat{R_P}$. This completes the proof of the Theorem. $\square$