Math 711: Lecture of October 26, 2007

It still remains to prove the final assertion of the Theorem from p. 3 of the Lecture Notes of October 22: that if R is F-finite and weakly F-regular, then R is strongly F-regular. Before doing so, we want to note some consequences of the theory of test elements, and also of the theory of approximately Gorenstein rings.

Theorem. *Let $(R,\, m,\, K)$ be a local ring of prime characteristic $p > 0$.*

(a) *If R has a completely stable test element, then $\widehat{R}$ is weakly F-regular if and only if R is weakly F-regular.*

(b) *If R has a completely stable big test element, then $\widehat{R}$ has the property that every submodule of every module is tightly closed if and only if R does.*

Proof. We already know that if a faithfully flat extension has the relevant property, then R does. For the converse, it suffices to check that 0 is tightly closed in every finite length module over $\widehat{R}$ (respectively, in the injective hull E of the residue class field over $\widehat{R}$, which is the same as the injective hull of the residue class field over R). A finite length $\widehat{R}$-module is the same as a finite length R-module. We can use the completely stable (big, for part (b)) test element $c \in R$ in both tests, which are then bound to have the same outcome for each element of the modules. For a module M supported only at m,

$$\mathcal{F}^e_{\widehat{R}}(M) \cong \mathcal{F}^e_{\widehat{R}}(\widehat{R} \otimes_R M) \cong \widehat{R} \otimes_R \mathcal{F}^e_R(M) \cong \mathcal{F}^e_R(M). \qquad \square$$

Proposition. *Let R have a test element (respectively, a big test element) c and let $N \subseteq M$ be finitely generated (respectively, arbitrary) R-modules. Let $d \in R^\circ$ and suppose $u \in M$ is such that $cu^q \in N^{[q]}$ for infinitely many values of q. Then $u \in N^*_M$.*

Proof. Suppose that $du^q \in N^{[q]}$ and that $p^{e_1} = q_1 < q$, so that $q = q_1 q_2$. Then $(du^{q_1})^{q_2} = d^{q_2-1}du^q \in (N^{[q_1]})^{[q_2]} = N^{[q]}$, and it follows that for all q_3, $(du^{q_1})^{q_2 q_3} \in (N^{[q_1]})^{[q_2 q_3]}$. Hence, $du^{q_1} \in (N^{[q_1]})^*$ in $\mathcal{F}^{e_1}(M)$ whenever $q_1 \leq q$. Hence, if $du^q \in N^{[q]}$ for arbitrarily large values of q, then $du^q \in (N^{[q]})^*$ in $\mathcal{F}^e(M)$ for all q and it follows that $cdu^q \in N^{[q]}$ for all q, so that $u \in N^*_M$. $\square$

Theorem. *Let R be a Noetherian ring of prime characteristic $p > 0$.*

(a) *If every ideal of R is tightly closed, then R is weakly F-regular.*

(b) *If R is local and $\{I_t\}_t$ is a descending sequence of irreducible m-primary ideals cofinal with the powers of m, then R is weakly F-regular if and only if I_t is tightly closed for all $t \geq 1$.*

Proof. (a) We already know that every ideal is tightly closed if and only if every ideal primary to a maximal ideal is tightly closed, and this is not affected by localization at a maximal ideal. Therefore, we may reduce to the case where R is local. The condition that every ideal is tightly closed implies that R is normal and, hence, approximately Gorenstein. Therefore, it suffices to prove (b). For (b), we already know that R is weakly F-regular if and only if 0 is tightly closed in every finitely generated R-module that is an essential extension of K. Such a module is killed by I_t for some $t \gg 0$, and so embeds in $E_{R/I_t}(K) \cong R/I_t$ for some t. Since I_t is tightly closed in R, 0 is tightly closed in R/I_t, and the result follows. $\square$

We next want to establish a result that will enable us to prove the final assertion of the Theorem from p. 3 of the Lecture Notes of October 22.

Theorem. *Let (R, m, K) be a complete local ring of prime characteristic $p > 0$. If R is reduced and $c \in R^\circ$, let $\theta_{q,c} : R \to R^{1/q}$ denote the R-linear map such that $1 \mapsto c^{1/q}$. Then the following conditions are equivalent:*

(1) *Every submodule of every module is tightly closed.*

(2) *0 is tightly closed in the injective hull $E = E_R(K)$ of the residue class field $K = R/m$ of R.*

(3) *R is reduced, and for every $c \in R^\circ$, there exists q such that the $\theta_{q,c}$ splits.*

(4) *R is reduced, and for some c that has a power which is a big test element for R, there exists q such that $\theta_{q,c}$ splits.*

(5) *R is reduced, and for some c such that R_c is regular, there exists q such that $\theta_{q,c}$ splits.*

Proof. Note that all of the conditions imply that R is reduced.

We already know that conditions (1) and (2) are equivalent. Let u denote a socle generator in E. Then we have an injection $K \to E$ that sends $1 \mapsto u$, and we know that 0 is tightly closed in E if and only if u is in the tight closure of 0 in E. This is the case if and only if for some $c \in R^\circ$ (respectively, for a single big test element $c \in R^\circ$), $cu^q = 0$ in $\mathcal{F}^e(E)$ for all $q \gg 0$. We may view $\mathcal{F}^e : R \to R$ as $R \subseteq R^{1/q}$ instead. Then $\mathcal{F}^e(E)$ is identified with $R^{1/q} \otimes_R E$, and R acts via the isomorphism $R \cong R^{1/q}$ such that $r \mapsto r^{1/q}$. Then u^q corresponds to $1 \otimes u$, and cu^q corresponds to $c^{1/q} \otimes u$.

Then $u \in 0^*_E$ if and only if for every $c \in R^\circ$ (respectively, for a single big test element $c \in R^\circ$), the map $K \to R^{1/q} \otimes_R E$ that sends $1 \mapsto c^{1/q} \otimes u$ is 0 for all $q \gg 0$. We may now apply the functor $\mathrm{Hom}_R(\_\,, E)$ to obtain a dual condition. Namely, $u \in 0^*_E$ if and only if for every $c \in R^0$ (respectively, for a single big test element $c \in R^\circ$), the map

$$\mathrm{Hom}_R(R^{1/q} \otimes_R E,\, E) \to \mathrm{Hom}_R(K,\, E)$$

is 0 for all $q \gg 0$. The map is induced by composition with $K \to R^{1/q} \otimes_R E$. By the adjointness of tensor and Hom, we may identify this map with

$$\mathrm{Hom}_R\bigl(R^{1/q},\, \mathrm{Hom}_R(E,\, E)\bigr) \to \mathrm{Hom}_R(K,\, E).$$

This map sends f to the composition of $K \to R^{1/q} \otimes_R E$ with the map such that $s \otimes v \mapsto f(s)(v)$. Since $\mathrm{Hom}_R(E,\, E) \cong R$ by Matlis duality and $\mathrm{Hom}_R(K,\, E) \cong K$, we obtain the map

$$\mathrm{Hom}_R(R^{1/q},\, R) \to K$$

that sends f to the image of $f(c^{1/q})$ in R/m.

Thus, $u \in 0^*_E$ if and only if for every $c \in R^0$ (respectively, for a single big test element $c \in R^\circ$), every $f : R^{1/q} \to R$ sends $c^{1/q}$ into m for every $q \gg 0$. This is equivalent to the statement that $\theta_{q,c} : R \to R^{1/q}$ sending $1 \to c^{1/q}$ does not split for every $q \gg 0$, since if $f(c^{1/q}) = a$ is a unit of R, $a^{-1}f$ is a splitting.

Note that if $R \to R^{1/q}$ sending $1 \mapsto c^{1/q}$ splits, then $R \to R^{1/q}$ splits as well: the argument in the Lecture Notes from September 21 (see pages 4 and 5) applies without any modification whatsoever. Moreover, the second Proposition on p. 5 of those notes shows that if one has the splitting for a given q, one also has it for every larger q.

We have now shown that $u \in 0^*_E$ if and only if for every $c \in R^0$ (respectively, for a single big test element $c \in R^\circ$), $\theta_{q,c} : R \to R^{1/q}$ sending $1 \to c^{1/q}$ does not split for every q.

Hence, 0 is tightly closed in E if and only if for every $c \in R^0$ (respectively, for a single big test element $c \in R^\circ$) the map $\theta_{q,c}$ splits for some q.

We have now shown that conditions (1), (2), and (3) are equivalent, and that (4) is equivalent as well provided that c is a big test element.

Now suppose that we only know that c has a power that is a big test element. Then this is also true for any larger power, and so we can choose $q_1 = p^{e_1}$ such that c^{q_1} is a test element. If the equivalent conditions (1) , (2), and (3) hold, then we also know that the map $R \to R^{1/qq_1}$ sending $1 \mapsto (c^{q_1})^{1/qq_1} = c^{1/q}$ splits for all $q \gg 0$, and we may restrict this splitting to $R^{1/q}$. Thus, (1) through (4) are equivalent.

Finally, (5) is equivalent as well, because we know that if $c \in R^\circ$ is such that R_c is regular, then c has a power that is a big test element. $\square$

Remark. It is not really necessary to assume that R is reduced in the last three conditons. We can work with $R^{(e)}$ instead of $R^{1/q}$, where $R^{(e)}$ denotes R viewed as an R-algebra via the structural homomorphism $\mathcal{F}^e$. We may then define $\theta_{q,c}$ to be the R-linear map $R \to R^{(e)}$ such that $1 \mapsto c$. The fact that this map is split for some some $c \in R^\circ$ and some q implies that R is reduced: if r is a nonzero nilpotent, we can replace it by a power which is nonzero but whose square is 0. But then the image of r is $r^q c = 0$, and the map is not even injective, a contradiction. Once we know that R is reduced, we can identify $R^{(e)}$ with $R^{1/q}$ and c is identified with $c^{1/q}$.

We want to apply the preceding Theorem to the F-finite case. We first observe:

Lemma. *Let $(R,\, m,\, K)$ be an F-finite reduce local ring. Then $\widehat{R}^{1/q} \cong \widehat{R^{1/q}} \cong \widehat{R} \otimes_R R^{1/q}$ for all $q = p^e$.*

Proof. $R^{1/q}$ is a local ring module-finite over R. Hence, the maximal ideal of R expands to an ideal primary to the maximal ideal of $R^{1/q}$, and it follows that $\widehat{R^{1/q}}$ is the $mR^{1/q}$-adic completion of $R^{1/q}$. Thus, we have an isomorphism $\alpha : \widehat{R^{1/q}} \cong \widehat{R} \otimes_R R^{1/q}$. Since R is reduced, so is $R^{1/q}$. Since R is F-finite, so is $R^{1/q}$, and $R^{1/q}$ is consequently excellent. Hence, the completion $\widehat{R^{1/q}}$ is reduced. If we use the identification α to write a typical element of $u \in \widehat{R^{1/q}}$ as a sum of terms of the form $s \otimes r^{1/q}$, where $s \in \widehat{R}$ and $r \in R$, we see that $u^q \in \widehat{R}$. This shows that we have $\widehat{R^{1/q}} \subseteq \widehat{R}^{1/q}$. On the other hand, if $r_0, r_1, \ldots, r_k, \ldots$ is a Cauchy sequence in R with limit s, then $r_0^{1/q}, r_1^{1/q}, \cdots, r_k^{1/q}, \cdots$ is a Cauchy sequence in $R^{1/q}$, and its limit is $s^{1/q}$. This shows that $\widehat{R}^{1/q} \subseteq \widehat{R^{1/q}}$. $\square$

From the preceding Theorem we then have:

Corollary. *If R is F-finite, then R is strongly F-regular if and only if every submodule of every module is tightly closed.*

Proof. We need only show that if every submodule of every module is tightly closed, then R is strongly F-regular. We know that both conditions are local on the maximal ideals of R (cf. problem 6. of Problem Set #3). Thus, we may assume that (R, m, K) is local. We know that R has a completely stable big test element c. By part (b) of the Theorem on the first page, $\widehat{R}$ has the property that every submodule of every module is tightly closed: in particular, 0 is tightly closed in $E = E_{\widehat{R}}(K) \cong E_R(K)$. By the equivalence of (2) and (4) in the preceding Theorem, we have that the $\widehat{R}$-linear map $\widehat{\theta} : \widehat{R} \to \widehat{R^{1/q}}$ that sends $1 \mapsto c^{1/q}$ splits for some q. This map arises from the R-linear map $\theta : R \to R^{1/q}$ that sends $1 \mapsto c^{1/q}$ by applying $\widehat{R} \otimes_R \_$. Since $\widehat{R}$ is faithfully flat over R, the map θ is split if and only if $\widehat{\theta}$ is split, and so θ is split as well. $\square$

Finally, we can prove the final statement in the Theorem on p. 4 of the Lecture Notes from October 22.

Corollary. *If R is Gorenstein and F-finite, then R is weakly F-regular if and only if R is strongly F-regular.*

Proof. The issue is local on the maximal ideals of R. We have already shown that in the local Gorenstein case, (R, m, K) is weakly F-regular if and only if 0 is tightly closed in $E_R(K)$. By the Corollary just above, this implies that R is strongly F-regular in the F-finite case. $\square$

This justifies extending the notion of *strongly F-regular* ring as follows: the definition agrees with the one given earlier if the ring is F-finite.

Definition. Let R be a Noetherian ring of prime characteristic $p > 0$. We define R to be *strongly F-regular* if every submodule of every module (whether finitely generated or not) is tightly closed.