

Math 711: Lecture of October 29, 2007

We want to study flat local homomorphism $(R, m, K) \rightarrow (S, \mathfrak{n}, L)$ to obtain information about S from corresponding information about the base R and the closed fiber S/mS . We recall one fact of this type from p. 3 of the Lecture Notes from September 19:

Proposition. *Let $(R, m, K) \rightarrow (S, \mathfrak{n}, L)$ be a flat local homomorphism of local rings. Then*

- (a) $\dim(S) = \dim(R) + \dim(S/mS)$, the sum of the dimensions of the base and of the closed fiber.
- (b) If R is regular and S/mS is regular, then S is regular.

Part (a) is generalized in the Theorem given below. We first want to note the following.

Lemma. *Let R a ring, and S a flat R -algebra.*

- (a) *If M is a finitely generated R -module, then*

$$\text{Ann}_S(S \otimes_R M) = (\text{Ann}_R M)S \cong S \otimes (\text{Ann}_R M).$$

Hence, if $N, N' \subseteq W$ are R -modules with N' finitely generated, $S \otimes_R N :_S S \otimes_R N' = (N :_R N')S$.

- (b) *If I is a finitely generated ideal of R , M is any R -module, and N is a flat R -module, then then*

$$\text{Ann}_{M \otimes_R N} I \cong (\text{Ann}_M I) \otimes_R N.$$

Proof. (a) Let u_1, \dots, u_n be generators of M . Then there is a map $\alpha : R \rightarrow M^n$ that sends $r \mapsto (ru_1, \dots, ru_n)$. The kernel of this map is evidently $\text{Ann}_R M$, so that

$$0 \rightarrow \text{Ann}_R M \rightarrow R \xrightarrow{\alpha} M^n$$

is exact. Since S is flat, we may apply $S \otimes_R _$ to get an exact sequence:

$$(*) \quad 0 \rightarrow S \otimes_R (\text{Ann}_R M) \rightarrow S \xrightarrow{\beta} (S \otimes_R M)^n$$

where the map $\beta = \mathbf{1}_S \otimes \alpha$ sends $s \in S$ to $s(1 \otimes u_1, \dots, 1 \otimes u_n)$. Since the elements $1 \otimes u_j$ generate $S \otimes_R M$, the kernel of β is $\text{Ann}_S(S \otimes_R M)$, while the kernel is also $S \otimes_R (\text{Ann}_R M)$ by the exactness of the sequence (*). For the final statement, one may simply observe that $N :_R N' = \text{Ann}_R(N + N')/N$.

(b) Let f_1, \dots, f_n be generators for I , and consider the map $\gamma : M \rightarrow M^n$ such that $u \mapsto (f_1 u, \dots, f_n u)$. Then

$$0 \rightarrow \text{Ann}_M I \rightarrow M \xrightarrow{\gamma} M^n$$

is exact. It follows that

$$(**) \quad 0 \rightarrow (\text{Ann}_M I) \otimes_R N \rightarrow M \otimes_R N \xrightarrow{\delta} (M \otimes_R N)^n$$

is exact, where $\delta = \gamma \otimes \mathbf{1}_N$ sends $u \otimes v \mapsto (f_1 u \otimes v, \dots, f_n u \otimes v)$. Therefore the kernel is $\text{Ann}_{M \otimes_R N} I$, but from $(**)$ the kernel is also $(\text{Ann}_M I) \otimes_R N$.

□

Comparison of depths and types for flat local extensions

In the Theorem below, the main case is when $N = S$, and the reader should keep this case in mind. When N is an S -module, we refer to N/mN as the *closed fiber* of N over R . When we refer to the *depth* of a finitely generated nonzero module over a local ring, we always mean the depth on the maximal ideal of the local ring, unless otherwise specified.

Theorem. *Let $(R, \mathfrak{m}, K) \rightarrow (S, \mathfrak{n}, L)$ be a local homomorphism of local rings, and let N be a finitely generated nonzero S -module that is R -flat. Let M be a nonzero finitely generated R -module. Then:*

(a) $\dim(N) = \dim(R) + \dim(N/mN)$.

More generally, $\dim(M \otimes_R N) = \dim(M) + \dim(N/mN)$.

(b) N is faithfully flat over R .

(c) *If $y_1, \dots, y_k \in \mathfrak{n}$ form a regular sequence on the closed fiber N/mN , then they form a regular sequence on $W \otimes_R N$ for every nonzero R -module W . In particular, they form a regular sequence on N/IN for every proper ideal I of R , and on N itself. Moreover, $N/(y_1, \dots, y_k)N$ is faithfully flat over R .*

(d) $\text{depth}(M \otimes_R N) = \text{depth}(M) + \text{depth}(N/mN)$. *In particular, $\text{depth}(N) = \text{depth}(R) + \text{depth}(N/mN)$.*

(e) N is Cohen-Macaulay if and only if both R and N/mN are Cohen-Macaulay.

More generally, if M is any finitely generated R -module, then $M \otimes_R N$ is Cohen-Macaulay if and only if both M and N/mN are Cohen-Macaulay.

(f) *If N is Cohen-Macaulay, then the type of N is the product of the type of R and the type of N/mN . More generally, if $M \otimes_R N$ is Cohen-Macaulay, the type of $M \otimes_R N$ is the product of the type of M and the type of N/mN .*

If M is 0-dimensional and N/mN is 0-dimensional, then the socle of $M \otimes_R N$ is the same as the socle of $\text{Ann}_M \mathfrak{m} \otimes_K N/mN$.

Proof. We use induction on $\dim(R)$ to prove the first statement. If J is the nilradical of R , we may make a base change and replace R , S , and N by R/J , S/JS , and N/JN . The

dimensions of R and N do not change, and the closed fiber does not change. Thus, we may assume that R is reduced. If $\dim(R) = 0$, then R is a field, $m = 0$, and the result is obvious. If $\dim(R) > 0$ then m contains a nonzerodivisor x , which is also a nonzerodivisor on N , because N is R -flat. We may make a base change to R/xR . The dimensions of R and N decrease by one, and the closed fiber does not change. Hence, the validity of the statement in (a) is not affected by this base change. The result now follows from the induction hypothesis.

To prove the second statement, note that M has a finite filtration whose factors are cyclic modules R/I_j , and so $M \otimes_R N$ has a corresponding filtration whose factors are the $(R/I_j) \otimes_R N$. Thus, we can reduce to the case where $M = R/I$. But this case follows from what we have already proved by base change from R to R/I .

(b) To see that a flat module N is faithfully flat over any ring R , it suffices to see that $Q \otimes_R N \neq 0$ for every nonzero R -module Q . If $u \in Q$ is a nonzero element, then $Ru \hookrightarrow Q$, and so it suffices to see that $Ru \otimes_R N \neq 0$ when $Ru \neq 0$. But $Ru \cong R/I$ for some proper ideal I of R , and $Ru \otimes_R N \cong Q/IQ$. Thus, it suffices to see that $IQ \neq Q$ for every proper ideal I of R . Since any proper ideal I is contained in a maximal ideal, it suffices to see that $mQ \neq Q$ when m is maximal in R . In the local case, there is only one maximal ideal to check. In the situation here, $mN \subseteq \mathfrak{n}N \neq N$ by Nakayama's Lemma, since N is finitely generated and nonzero.

(c) By a completely straightforward induction on k , part (c) reduces at once to the case where $k = 1$. (Note that an important element in making this inductive proof work is that we know that N/y_1N is again R -flat.)

Write y for y_1 . To see that N/yN is R -flat, it suffices by the local criterion for flatness, p. 1 of the Lecture Notes from October 1, to show that $\mathrm{Tor}_1^R(R/m, N/yN) = 0$. The short exact sequence

$$(*) \quad 0 \rightarrow N \xrightarrow{y} N \rightarrow N/yN \rightarrow 0$$

yields a long exact sequence part of which is

$$0 = \mathrm{Tor}_1^R(R/m, N) \rightarrow \mathrm{Tor}_1^R(R/m, N/yN) \rightarrow N/mN \xrightarrow{y} N/mN$$

where the 0 on the left is a consequence of the fact that N is R -flat. Since y is not a zerodivisor on N/mN , we have that $\mathrm{Tor}_1^R(R/m, N/yN) = 0$. Faithful flatness is then immediate from part (b).

Note that if $M \neq 0$, we have that $M \otimes_R (N/yN) \neq 0$ because N/yN is faithfully flat over R . To show that y is not a zerodivisor on $M \otimes_R N$, we may use the long exact sequence for Tor arising from the short exact sequence (*) displayed above when one applies $M \otimes_R _$. The desired result follows because $\mathrm{Tor}_1^R(M, N/yN) = 0$.

(d) We use induction on $\mathrm{depth}(M) + \mathrm{depth}(N/mN)$. Suppose that both depths are 0. Because $\mathrm{depth}(M) = 0$, m is an associated prime of M , and there is an injection $R/m \hookrightarrow M$. We may apply $_ \otimes_R N$ to obtain an injection $N/mN \hookrightarrow M \otimes_R N$. Since $\mathrm{depth}(N/mN) = 0$, there is an injection $S/\mathfrak{n} \hookrightarrow N/mN$, and the composite

$$S/\mathfrak{n} \hookrightarrow N/mN \hookrightarrow M \otimes_R N$$

shows that $\text{depth}(M \otimes_R N) = 0$, as required. If $\text{depth}(M) > 0$ we may choose $x \in m$ not a zerodivisor on M . Then x is not a zerodivisor on $M \otimes_R N$ because N is R -flat. We may replace M by M/xM . The depths of M and $M \otimes_R N$ decrease by 1, while the closed fiber does not change. The result now follows from the induction hypothesis. Finally, suppose $\text{depth}(N/mN) > 0$. Choose $y \in \mathfrak{n}$ not a zerodivisor on N/mN . Then we may replace N by N/yN . This module is still flat over R by part (c). The depths of N and N/mN decrease by one, while M is unaffected. Again, the result follows from the induction hypothesis.

(e) $M \otimes_R N$ is Cohen-Macaulay if and only if $\dim(M \otimes_R N) = \text{depth}(M \otimes_R N)$, and by parts (a) and (c), we have that

$$\begin{aligned} \dim(M \otimes_R N) - \text{depth}(M \otimes_R N) &= \dim(M) + \dim(N/mN) - (\text{depth}(M) + \text{depth}(N/mN)) \\ &= (\dim(M) - \text{depth}(M)) + (\dim(N/mN) - \text{depth}(N/mN)). \end{aligned}$$

The last sum is 0 if and only if both summands vanish, since both summands are nonnegative. This proves (e).

(f) Choose a maximal regular sequence $x_1, \dots, x_h \in m$ on M . This is also a regular sequence on $M \otimes_R N$, and so we may replace M by $M/(x_1, \dots, x_h)M$. Both types are unaffected by this replacement. Similarly, we may choose $y_1, \dots, y_k \in \mathfrak{n}$ that form a maximal regular sequence on N/mN , and replace N by $N/(y_1, \dots, y_k)N$. Hence, we have reduced to the case where M and N/mN both are 0-dimensional.

We next want to establish the final statement, from which the calculation of the type will follow. We have that $\text{Ann}_{\mathfrak{n}}(M \otimes_R N) \subseteq \text{Ann}_m(M \otimes_R N) \cong (\text{Ann}_M m) \otimes_R N$, by part (b) of the Lemma from p. 1. Since m kills the module $\text{Ann}_M m$, this tensor product may be identified with $(\text{Ann}_M m) \otimes_R (N/mN) \cong (\text{Ann}_M m) \otimes_K (N/mN)$. Then $\text{Ann}_M n \cong K^s$ where s is the type of M , and this module is $(N/mN)^{\oplus s}$. Clearly the socle is the direct sum of s copies of the socle in N/mN , whose dimension over L is the type of N/mN , and the statement about the product of the types is immediate. \square

Corollary. *Let $(R, m, K) \rightarrow (S, \mathfrak{n}, L)$ be a flat local homomorphism of local rings.*

- (a) *S is Gorenstein if and only if R and S/mS are both Gorenstein.*
- (b) *If I is an irreducible m -primary ideal in R and $y_1, \dots, y_k \in \mathfrak{n}$ have images that are a system of parameters in S/mS , then for every integer t , $IS + (y_1^t, \dots, y_k^t)S$ is an irreducible m -primary ideal of S .*

Proof. Part (a) is immediate from parts (e) and (f) of the preceding Theorem, since the type of S is the product of the types of R and S/mS , and so the type of S will be 1 if and only if and only if both R and S/mS are of type 1.

For part (b), make a base change to R/I , which is 0-dimensional Gorenstein ring. Thus, S/IS is Gorenstein, the images of y_1^k, \dots, y_k^k are a system of parameters and, hence, a maximal a regular sequence. Thus, the quotient of S/I by the ideal they generate is a 0-dimensional Gorenstein ring. \square

We can now prove:

Theorem. *Let $(R, m, K) \rightarrow (S, \mathfrak{n}, L)$ be a flat local homomorphism of local rings of prime characteristic $p > 0$ such that the closed fiber S/m is regular. Suppose that $c \in R^\circ$ is a test element for both R and S . If R is weakly F-regular, then S is weakly F-regular.*

Proof. It suffices to show that a sequence of irreducible ideals cofinal with the powers of \mathfrak{n} is tightly closed. Since R is weakly F-regular, it is normal, and, hence, approximately Gorenstein. Given t , choose $I \subseteq m^t$ that is m -primary and irreducible. Let $u \in m$ represent a socle generator in R/I . Choose y_1, \dots, y_k whose images in the regular local ring S/mS are a minimal set of generators of the maximal ideal. Then $(y_1 \cdots y_k)^{t-1}$ represents a socle generator in $(S/mS)/(y_1^t, \dots, y_k^t)(S/mS)$. It follows from part (f) of the preceding Theorem that

$$w_t = (y_1 \cdots y_k)^{t-1}u$$

represents a socle generator in

$$S/((IS + (y_1^t, \dots, y_k^t)S));$$

this ideal is irreducible by the preceding Corollary. Note that $IS + (y_1^t, \dots, y_k^t)S \subseteq \mathfrak{n}^t$. It will therefore suffice to prove that every ideal J of the form $IS + (y_1^t, \dots, y_k^t)S$ is tightly closed, and for this it suffices to check that w_t is not in the tight closure. Since c is a test element for both rings, if w_t were in the tight closure of J we would have

$$c((y_1 \cdots y_k)^{t-1}u)^q \in I^{[q]}S + (y_1^{qt}, \dots, y_k^{qt})S$$

for all $q \gg 0$. By part (c) of the Theorem on p. 2, y_1, \dots, y_k is regular sequence on $S/I^{[q]}S$. Hence, modulo $I^{[q]}S$, cu^q is in

$$(y_1^{qt}, \dots, y_k^{qt})(S/I^{[q]}S) :_{S/I^{[q]}S} y_1^{tq-q} \cdots y_k^{tq-q} = (y_1^q, \dots, y_k^q)(S/I^{[q]}S),$$

and so

$$cu^q \in I^qS + (y_1^q, \dots, y_k^q)S$$

for all $q \gg 0$. Now work modulo $(y_1^q, \dots, y_k^q)S$ instead. Since $T = S/(y_1^q, \dots, y_k^q)S$ is faithfully flat over R by the part (c) of the Theorem on p. 2, we may think of R as a subring of T . But then we have that $cu^q \in I^{[q]}T \cap R = I^{[q]}$ for all $q \gg 0$, and so $u \in I_R^* = I$, a contradiction, because u represents a socle generator in R/I . \square

We next want to prove a corresponding result for strong F-regularity: this works in just the same way, except that we need to assume that c is a big test element. However, the proof involves some understanding of the injective hull of the residue class field of S in this situation.