Math 711: Lecture of October 31, 2007

Discussion: local cohomology. Let $y_1, \ldots, y_d$ be a sequence of elements of a Noetherian ring S and let N be an S-module, which need not be finitely generated. Let J be an ideal whose radical is the same as the radical of $(y_1, \ldots, y_d)S$. Then the dth local cohomology module of N with supports in J, denoted $H^d_J(N)$, may be obtained as

$$\varinjlim_k \frac{N}{(y_1^k, \ldots, y_d^k)N}$$

where the map from

$$N_k = \frac{N}{(y_1^k, \ldots, y_d^k)N}$$

to N_{k+h} is induced by multiplication by z^h, where $z = y_1 \cdots y_d$, on the numerators.. If $u \in N$, $\langle u;\, y_1^k, \ldots, y_d^k\rangle$ denotes the image of the class of u in N_k in $H^d_J(N)$. With this notation, we have that

$$\langle u;\, y_1^k, \ldots, y_d^k\rangle = \langle z^h u;\, y_1^{k+h}, \ldots, y_d^{k+h}\rangle$$

for every $h \in \mathbb{N}$.

If $y_1, \ldots, y_d$ is a regular sequence on N, these maps are injective. We also know from the seminar that if $(S, \mathfrak{n}, L)$ is a Gorenstein local ring and $\underline{y} = y_1, \ldots, y_d$ is a system of parameters for S, then $H^d_{(\underline{y})}(S) = H^d_{\mathfrak{n}}(S)$ is an injective hull for the residue class field $L = S/\mathfrak{n}$ of S over S. In the sequel, we want to prove a relative form of this result when $R \to S$ is a flat local homomorphism whose closed fiber is Gorenstein.

Theorem. *Let $(R, m, K) \to (S, \mathfrak{n}, L)$ be a flat local homomorphism such that the closed fiber S/mS is Gorenstein. Let $\dim(R) = n$ and let $\dim(S/mS) = d$. Let $\underline{y} = y_1, \ldots, y_d \in \mathfrak{n}$ be elements whose images in S/mS are a system of parameters. Let $E = E_R(K)$ be an injective hull for the residue class field $K = R/m$ of R over R. Then $E \otimes_R H^d_{(\underline{y})}(S)$ is an injective hull for $L = S/\mathfrak{n}$ over S.*

In the case where the rings are of prime characteristic $p > 0$,

$$\mathcal{F}^e_S\big(E \otimes_R H^d_{(\underline{y})}(S)\big) \cong \mathcal{F}^e_R(E) \otimes_R H^d_{(\underline{y})}(S),$$

and if $u \in E$ and $s \in S$, then

$$(u \otimes \langle s;\, y_1^k, \ldots, y_d^k\rangle)^q = u^q \otimes \langle s^q;\, y_1^{qk}, \ldots, y_d^{qk}\rangle.$$

Proof. We first give an argument for the case where R is approximately Gorenstein, which is somewhat simpler. We then treat the general case. Suppose that $\{I_t\}$ is a descending sequence of m-primary ideals of R cofinal with the powers of M. We know that $E = \varinjlim_t R/I_t$ for any choice of injective maps $R/I_t \to R/I_{t+1}$. Let $\mathfrak{A}_{t,k} = I_t S + J_k$, where $J_k = (y_1^k, \ldots, y_d^k)S$. For every k we may tensor with the faithfully fflat R-algebra S/J_k to obtain an injective map $S/\mathfrak{A}_{t,k} \to S/\mathfrak{A}_{t+1,k}$. Since $y_1, \ldots, y_d$ is a regular sequence on $S/I_t S$ for every I_t, we also have an injective map $S/\mathfrak{A}_{t,k} \to S/\mathfrak{A}_{t,k+1}$ induced by multiplication by $z = y_1 \cdots y_d$ on the numerators. The ideals $\mathfrak{A}_{t,k}$ are $\mathfrak{n}$-prinary irreducible ideals and as t, k both become large, are contained in aribtrarily large powers of $\mathfrak{n}$. (Once $I_t \subseteq m^s$ and $k \geq s$, we have that $\mathfrak{A}_{t,\,k} \subseteq m^s S + \mathfrak{n}^s \subseteq \mathfrak{n}^s$.) Thus, we have

$$E_S(L) \cong \varinjlim_{t,k} \frac{S}{\mathfrak{A}_{t,k}} = \varinjlim_{t,k} \left(\frac{R}{I_t} \otimes_R \frac{S}{J_k}\right) \cong \varinjlim_k \left(\varinjlim_t \left(\frac{R}{I_t} \otimes_R \frac{S}{J_k}\right)\right) \cong$$

$$\varinjlim_k \left(\left(\varinjlim_t \frac{R}{I_t}\right) \otimes_R \frac{S}{J_k}\right) \cong \varinjlim_k \left(E \otimes_R \frac{S}{J_k}\right) \cong E \otimes_R \left(\varinjlim_k \frac{S}{J_t}\right) \cong E \otimes_R H^d_{(\underline{y})}(S).$$

We now give an alternative argument that works more generally. In particular, we do not assume that R is approximately Gorenstein. Let E_t denote $\mathrm{Ann}_E m^t$. We first claim that that $E_{t,k}$, which we define as $E_t \otimes_R (S/J_k)$, is an injective hull of L over $S_{t,k} = (R/m^t) \otimes_R (S/J_k)$. By part (f) of the Theorem on p. 2 of the Lecture Notes from October 29, it is Cohen-Macaulay of type 1, since that is true for E_t and for the closed fiber of S/J_k, since S/mS is Gorenstein. Hence, $E_{t,k}$ is an essential extension of L, and it is killed by $\mathfrak{A}_{t,k} = m^t S + J_k$. To complete the proof, it suffices to show that it has the same length as $S_{t,k}$. Let M denote either R/m^t or E_t. Note that M has a filtration with $\ell(M)$ factors, each of which is $\cong K = R/m$. Since S/J_k is R-flat, this gives a filtration of $M \otimes_R S/J_k$ with $\ell(M)$ factors each of which is isomorphic with $K \otimes_R S/J_k = S/(mS + J_k)$. Since $\ell(R/m^t) = \ell(E_t)$, it follows that $S_{t,k}$ and $E_{t,k}$ have the same length, as required.

If $t \leq t'$ we have an inclusion $E_t \hookrightarrow E_{t'}$, and if $k \leq k'$, we have an injection $S/J_k \to S/J_{k'}$ induced by multplication by $z^{k'-k}$ acting on the numerators. This gives injections $E_t \otimes_R S_k \to E_{t'} \otimes S_k$ (since S_k is R-flat) and $E_{t'} \otimes_R S_k \to E_{t'} \otimes_R S'_k$ (since $y_1, \ldots, y_d$ is a regular sequence on $E_{t'} \otimes_R S$). The composites give injections $E_{t,k} \hookrightarrow E_{t',k'}$ and the direct limit over t, k is evidently $E \otimes_K H^d_{(\underline{y})}(S)$. The resulting module is clearly an essential extension of L, since it is a directed union of essential extensions. Hence, it is contained in a maximal essential extension $E_S(L)$ of L over S. We claim that this inclusion is an equality. To see this, suppose that $u \in E_S(L)$ is any element. Then u is killed by $\mathfrak{A} = \mathfrak{A}_{t,k} = m^t S + J_k$ for any sufficiently large choices of t and k. Hence $u \in \mathrm{Ann}_{E_S(L)} \mathfrak{A} = N$, which we know is an injective hull for L over $S/\mathfrak{A}$. But $E_t \otimes_R S/J_k$ is a submodule of N contained in $E \otimes_R H^d_{(\underline{y})}(S)$, and is already an injective hull for L over $S/\mathfrak{A}$. It follows, since they have the same length, that we must have that $E_t \otimes_R S/J_k \subseteq N$ is all of N, and so $u \in E_t \otimes_R S/J_k \subseteq E \otimes_R H^d_{(\underline{y})}(S)$.

To prove the final statement about the Frobenius functor, we note that by the first problem of Problem Set #4, one need only calculate $\mathcal{F}^e_S\big(H^d_{(\underline{y})}(S)\big)$, and this calculation is the precisely the same as in third paragraph of p. 1 of the Lecture Notes from October 24. $\square$

We are now ready to prove the analogue for strong F-regularity of the Theorem at the top of p. 5 of the Lecture Notes from October 29, which treated the weakly F-regular case.

Theorem. *Let $(R, m, K) \to (S, \mathfrak{n}, L)$ be a local homomorphism of local rings of prime characteristic $p > 0$ such that the closed fiber S/m is regular. Suppose that $c \in R^\circ$ is a big test element for both R and S. If R is strongly F-regular, then S is strongly F-regular.*

Proof. Let u be a socle generator in $E = E_R(K)$, and let $\underline{y} = y_1, \dots, y_d \in \mathfrak{n}$ be elements whose images in the closed fiber S/mS form a minimal set of generators of the maximal ideal $\mathfrak{n}/mS$. Let $z = y_1 \cdots y_d$. Then the image of 1 in $S/\bigl(mS + (y_1, \dots, y_d)S\bigr)$ is a socle generator, and it follows that $v = u \otimes \langle 1; y_1, \dots, y_d\rangle$ generates the socle in $E_S(L) \cong E \otimes_R H^d_{(\underline{y})}(S)$. Since c is a big test element for S, it can be used to test whether v is in the tight closure of 0 in $E \otimes_R H^d_{(\underline{y})}(S)$.

This occurs if and only if for all $q \gg 0$, $c(u \otimes \langle 1; y_1, \dots, y_d\rangle)^q = 0$ in $\mathcal{F}^e_R(E) \otimes_R H^d_{(\underline{y})}(S)$, and this means that $cu^q \otimes \langle 1; y_1^q, \dots, y_d^q\rangle = 0$ in $\mathcal{F}^e_R(E) \otimes_R H^d_{(\underline{y})}(S)$. By part (c) of the Theorem on p. 2 of the Lecture Notes from October 29, $y_1, \dots, y_d$ is a regular sequence on $E \otimes_R S$, from which it follows that the module $\mathcal{F}^e_R(E) \otimes_R \bigl(S/(y_1^q, \dots, y_d^q)\bigr)$ injects into $\mathcal{F}^e_R(E) \otimes_R H^d_{(\underline{y})}(S)$. Since $\overline{S} = S/(y_1^q, \dots, y_d^q)S$ is faithfully flat over R, the map $\mathcal{F}^e_R(E) \to \mathcal{F}^e_R(E) \otimes_R S/(y_1^q, \dots, y_d^q)S$ sending $w \mapsto w \otimes 1$ is injective. The fact that $cu^q \otimes \langle 1; y_1^q, \dots, y_d^q\rangle = 0$ implies that $cu^q \otimes 1_{\overline{S}}$ is 0 in $\mathcal{F}^e_R(E) \otimes_R \overline{S}$, and hence that $cu^q = 0$ in R. Since this holds for all $q \gg 0$, we have that $u \in 0^*_E$, a contradiction. $\square$

The following result will be useful in studying algebras essentially of finite type over an excellent semilocal ring that are not F-finite but are strongly F-regular: in many instances, it permits reductions to the F-finite case.

Theorem. *Let R be a reduced Noetherian ring of prime characteristic $p > 0$ that is essentially of finite type over an excellent semilocal ring B.*

(a) *Let $\widehat{B}$ denote the completion of B with respect to its Jacobson radical. Suppose that R is strongly F-regular. Then $\widehat{B} \otimes_B R$ is essentially of finite type over $\widehat{B}$ and is strongly F-regular and faithfully flat over R.*

(b) *Suppose that $B = A$ is a complete local ring with coefficient field K. Fix a p-base Λ for K. For all $\Gamma \ll \Lambda$, let $R^\Gamma = A^\Gamma \otimes_A R$. We nay identify $\mathrm{Spec}\,(R^\Gamma)$ with $X = \mathrm{Spec}\,(R)$ as topological spaces, and we let Z_Γ denote the closed set in $\mathrm{Spec}\,(R)$ of points corresponding to primes P such that R^Γ_P is not strongly F-regular. Then Z_Γ is the same for all sufficiently small $\Gamma \ll \Lambda$, and this closed set is the locus in X consisting of primes P such that R_P is not strongly F-regular.*

In particular, if R is strongly F-regular, then for all $\Gamma \ll \Lambda$, R^Γ is strongly F-regular.

Proof. (a) Since $B \to \widehat{B}$ is faithfully flat with geometrically regular fibers, the same is true for $R \to \widehat{B} \otimes_B R$. Choose $c \in R^\circ$ such that R_c is regular. Then we also have that

$(\widehat{B} \otimes_B R)_c$ is regular. Hence, c has a power that is a completely stable big test element in both rings. Let Q be any prime ideal of $S = \widehat{B} \otimes_B R$ and let P be its contraction to R. We may apply the preceding Theorem to the map $R_P \to S_Q$, and so S_Q is strongly F-regular for all Q. It follows that S is strongly F-regular.

(b) For all choices of $\Gamma' \subseteq \Gamma$ cofinite in Λ, we have that $R \subseteq R^{\Gamma'} \subseteq R^{\Gamma}$, and that the maps are faithfully flat and purely inseparable. Since every R^{Γ} is F-finite, we know that every Z_{Γ} is closed. Since the map $R^{\Gamma'} \subseteq R^{\Gamma}$ is faithfully flat, Z_{Γ} decreases as Γ decreases. We may choose Γ so that $Z = Z_{\Gamma}$ is minimal, and, hence, minimum, since a finite intersection of cofinite subsets of Λ is cofinite.

We shall show that Z must be the set of primes P in $\mathrm{Spec}\,(R)$ such that R_P is strongly F-regular. If Q is a prime of R^{Γ} not in Z_{Γ} lying over P in R, the fact that $R_P \to R^{\Gamma}_Q$ is faithfully flat implies that P is not in Z. Thus, $Z \subseteq Z_{\Gamma}$. If they are not equal, then there is a prime P of R such that R_P is strongly F-regular but R^{Γ}_Q is not strongly F-regular, where Q is the prime of R^{Γ} corresponding to P. Choose $\Gamma' \subseteq \Gamma$ such that $Q' = PR^{\Gamma'}$ is prime. It will suffice to prove that $S = R^{\Gamma'}_{Q'}$ is strongly F-regular, for this shows that $Z_{\Gamma'} \subseteq Z_{\Gamma} - \{P\}$ is strictly smaller than Z_{Γ}. Since S is F-finite, we may choose a big test element c_1 for S. Then c_1 has a q_1 th power c in R_P for some q_1, and c is still a big test element for S. The closed fiber of $R_P \to S$ is $S/PS = S/Q'$, a field. Hence, by the preceding Theorem, S is strongly F-regular. $\square$

Using this result, we can now prove:

Theorem. *Let R be reduced and essentially of finite type over an excellent semilocal ring B. Then the strongly F-regular locus in R is Zariski open.*

Proof. We first consider the case where $B = A$ is a complete local ring. Choose a coefficient field K for A and a p-base Λ for it. Then the result is immediate from part (b) of the preceding Theorem by comparison with R^{Γ} for any $\Gamma \ll \Lambda$.

In the general case, let $S = \widehat{B} \otimes_B R$. Since $\widehat{B}$ is a finite product of complete local rings, S is a finite product of algebras essentially of finite type over a complete local ring, and so the non-strongly F-regular locus is closed. Let J denote an ideal of S that defines this locus.

Now consider any prime ideal P of R such that R_P is strongly F-regular. Let $W = R - P$. Then we may apply part (a) of the preceding Theorem to $R_P \to \widehat{B} \otimes_B R_P$ to conclude that $\widehat{B} \otimes_R R_P = W^{-1}S$ is strongly F-regular. It follows that W must meet J: otherwise, we can choose a prime Q of S containing J but disjoint from W, and it would follow that S_Q is strongly F-regular even though $J \subseteq Q$, a contradiction. Choose $c \in W \cap J$. Then S_c is strongly F-regular, and since $R_c \to S_c$ is faithfully flat, so is R_c. Thus, the set of primes of R not containing c is a Zariski open nieghborhood of P that is contained in the strongly F-regular locus. $\square$