

Math 711: Lecture of November 5, 2007

The following result is one we have already established in the F-finite case. We can now extend it to include rings essentially of finite type over an excellent semilocal ring.

Theorem. *Let R be a reduced ring of prime characteristic $p > 0$ essentially of finite type over an excellent semilocal ring B . Suppose that $c \in R^\circ$ is such that R_c is strongly F-regular. Then c has a power that is a completely stable big test element in R .*

Proof. If c is a completely stable big test element in a faithfully flat extension of R , then that is also true for R by part (b) of the Proposition at the bottom of p. 8 of the Lecture Notes from September 17.

The hypothesis continues to hold if we replace R by $\widehat{B} \otimes_B R$, and it holds in each factor of this ring. We may therefore assume that R is essentially of finite type over a complete local ring A . As usual, choose a coefficient field K for A and a p -base Λ for K . Again, the hypothesis continues to hold if we replace R by R^Γ for $\Gamma \ll \Lambda$, and R^Γ is faithfully flat over R . But now we are done, since R^Γ is F-finite. \square

We next want to backtrack and prove that certain rings are approximately Gorenstein in a much simpler way than in the lengthy and convoluted argument given in the Lecture Notes from October 24. While the result we prove is much weaker, it does suffice for the case of an excellent normal Cohen-Macaulay ring, and, hence, for excellent weakly F-regular rings.

We first note:

Lemma. *Let M and N be modules over a Noetherian ring R and let x be a nonzerodivisor on N . Suppose that M is R -free or, much more generally, that $\text{Ext}_R^1(M, N) = 0$. Then*

$$(R/xR) \otimes_R \text{Hom}_R(M, N) \cong \text{Hom}_{R/xR}(M/xM, N/xN).$$

Proof. The right hand module is evidently the same as $\text{Hom}_R(M/xM, N/xN)$, and also the same as $\text{Hom}_R(M, N/xN)$, since any map $M \rightarrow N/xN$ must kill xM . Apply $\text{Hom}_R(M, _)$ to the short exact sequence

$$0 \rightarrow N \xrightarrow{x} N \rightarrow N/xN \rightarrow 0.$$

This yields a long exact sequence which is, in part,

$$0 \rightarrow \text{Hom}_R(M, N) \xrightarrow{x} \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M, N/xN) \rightarrow \text{Ext}_R^1(M, N) = 0,$$

and the result follows. \square

Theorem. *Let (R, m, K) be an excellent, normal, Cohen-Macaulay ring, or, more generally, any Cohen-Macaulay local ring whose completion is a Cohen-Macaulay local domain. Then R is approximately Gorenstein.*

Proof. We may replace R by its completion and then R is module-finite over a regular local ring $A \subseteq R$. Because R is Cohen-Macaulay, it is free of some rank h as an A -module, i.e., $R \cong A^h$. Then $\omega = \text{Hom}_A(R, A)$ is also an R -module, and is also isomorphic to A^h as an A -module. (We shall see later that ω is what is called a *canonical module* for R . Up to isomorphism, it is independent of the choice of A .) Then ω is, evidently, also a Cohen-Macaulay module over R . We want to see that it has type one. This only uses the Cohen-Macaulay property of R : it does not use the fact that R is a domain.

From the Lemma above, we see that the calculation of ω commutes with killing a parameter in A . We may choose a system of parameters for A (and R) that is a minimal set of generators for the maximal ideal of A . By killing these one at a time, we reduce to seeing this when $A = K$ is a field and R is a zero-dimensional local ring with coefficient field K . We claim that in this case, $\omega = \text{Hom}_K(R, K)$ is isomorphic with $E_R(K)$. In fact, ω is injective because for any R -module M ,

$$\text{Hom}_R(M, \omega) \cong \text{Hom}_R(M, \text{Hom}_K(R, K)) \cong \text{Hom}_K(M \otimes_R R, K) \cong \text{Hom}_K(M, K)$$

by the adjointness of tensor and Hom. This is, in fact, a natural isomorphism of functors. Since $\text{Hom}_K(_, K)$ is exact, so is $\text{Hom}_R(_, \omega)$. Thus, ω is a direct sum of copies of $E_R(K)$. But its length is the same as its dimension as a K -vector space, and this is the same as the dimension of R as a K -vector space, which is the length of R . Thus, ω has the same length as $E_R(K)$, and it follows that $\omega \cong E_R(K)$.

We now return to the situation where R is a domain. Since every nonzero element of R has a nonzero multiple in A , we have that ω is torsion-free as an R -module. Thus, if w is any nonzero element of ω , we have an embedding $R \rightarrow \omega$ sending $1 \mapsto w$. Let $I_t = (x_1^t, \dots, x_n^t)$, where x_1, \dots, x_n is a system of parameters for R . Then $I_t \omega \cap R w$ must have the form $J_t w$ for some m -primary ideal J_t of R . Then

$$R/J_t \cong R w / (I_t \omega \cap R w) \subseteq \omega / I_t \omega.$$

Since $\omega / I_t \omega$ is an injective hull of the residue class field for R/I_t , it is an essential extension of its socle. Therefore, R/J_t is an essential extension of its socle as well. Consequently, $J_t \subseteq R$ is irreducible and m -primary. It will now suffice to show that the ideals J_t are cofinal with the powers of m .

By the Artin-Rees Lemma there exists a constant integer $a \in \mathbb{N}$ such that

$$m^{N+a} \omega \cap R w \subseteq m^N (R w) = m^N w$$

for all N . But then $J_{N+a} \subseteq m^N$, since $I_{N+a} \subseteq m^{N+a}$. \square

We next want to prove some additional results on openness of loci, such as the Cohen-Macaulay locus. The following fact is very useful.

Lemma on openness of loci. *Let $X = \text{Spec}(R)$, where R is a Noetherian ring. Then $S \subseteq X$ is open if and only if the following two conditions hold:*

- (1) *If $P \subseteq Q$ and $Q \in S$ then $P \in S$.*
- (2) *For all $P \in S$, $S \cap \mathcal{V}(P)$ is open in $\mathcal{V}(P)$.*

The second condition can be weakened to:

- (2°) *For all $P \in S$, S contains an open neighborhood of P in $\mathcal{V}(P)$.*

Proof. It is clear that (1), (2), and (2°) are necessary for S to be open. Since (2°) is weaker than (2), and it suffices to show that (1) and (2°) imply that S is open. Suppose otherwise. Since R has DCC on prime ideals, if S is not open there exists a minimal element P of S that has no open neighborhood entirely contained in S . For all primes Q strictly contained in P , choose an open neighborhood U_Q of Q contained entirely in S . Let U be the union of these open sets: the U is an open set contained entirely in S , and contains all primes Q strictly smaller than P .

Let $Z = X - U$, which is closed. It follows that Z has finitely many minimal elements, one of which must be P . Call them $P = P_0, P_1, \dots, P_k$. Then

$$Z = \mathcal{V}(P_0) \cup \dots \cup \mathcal{V}(P_k).$$

Finally, choose U' open in X such that $P \in U'$ and $U' \cap \mathcal{V}(P) \subseteq S$. We claim that

$$U'' = U \cup U' - (\mathcal{V}(P_1) \cup \dots \cup \mathcal{V}(P_k))$$

is the required neighborhood of P . It is evidently an open set that contains P . Suppose that $Q \in U''$. If $Q \in U$ then $Q \in S$. Otherwise, Q is in

$$X - U = \mathcal{V}(P) \cup \mathcal{V}(P_1) \cup \dots \cup \mathcal{V}(P_k),$$

and this implies that $Q \in \mathcal{V}(P)$. But Q must also be in U' , and $U' \cap \mathcal{V}(P) \subseteq S$. \square

We can use this to show:

Theorem. *Let R be an excellent ring. Then the Cohen-Macaulay locus*

$$\{P \in \text{Spec}(R) : R_P \text{ is Cohen-Macaulay}\}$$

is Zariski open.

Proof. It suffices to establish (1) and (2) of the preceding Lemma. We know (1) because if $P \subseteq Q$ then R_P is a localization of the Cohen-Macaulay ring R_Q . Now suppose that R_P is Cohen-Macaulay. Choose a maximal regular sequence in PR_P . After multiplying by suitable units in R_P , we may assume that this regular sequence consists of images of elements $x_1, \dots, x_d \in P$. We can choose $c_i \in R - P$ that kills the annihilator of x_{i+1} in

$R/(x_1, \dots, x_i)$, $0 \leq i \leq d-1$. Let c be the product of the c_i . Then we may replace R by R_c (we may make finitely many such replacements, each of which amounts to taking a smaller Zariski open neighborhood of P).

Then x_1, \dots, x_d is a regular sequence in P , and is therefore a regular sequence in Q and in QR_Q for all primes $Q \supseteq P$. Hence, in considering property (2), it suffices to work with $R_1 = R/(x_1, \dots, x_d)R$: whether R_Q is Cohen-Macaulay or not is not affected by killing a regular sequence. Consequently, we need only show that if P is a minimal prime of an excellent ring R , then there exists $c \notin P$ such that R_c is Cohen-Macaulay. We may assume by localizing at one element in the other minimal primes but not in P that P is the only minimal prime of R .

First, we may localize at one element $c \notin P$ such that $(R/P)_c$ is a regular domain, because R/P is an excellent domain: the localization at the prime ideal (0) is a field, and, hence, regular, and so (0) has a Zariski open neighborhood that is regular. Henceforth, we assume that R/P is regular. It would suffice in the argument that follows to know that it is Cohen-Macaulay.

Finally, choose a filtration $P = P_1 \supseteq \dots \supseteq P_n = (0)$ such that every P_i/P_{i+1} is killed by P . We can do this because P is nilpotent. (If $P^n = 0$, we may take $P_i = P^i$, $1 \leq i \leq n$. Alternatively, we may take $P_i = \text{Ann}_P P^{n-i}$.) Second, we can localize at one element $c \in R - P$ such that each of the (R/P) -modules P_i/P_{i+1} , $1 \leq i \leq n-1$, is free over R/P . Here, we are writing R for the localized ring. We claim that the ring R is now Cohen-Macaulay.

To see this, suppose that we take any local ring of R . Then we may assume that (R, m, K) is local with unique minimal prime P , that R/P is Cohen-Macaulay, and that P has a finite filtration whose factors are free (R/P) -modules: all this is preserved by localization. Let x_1, \dots, x_h be a system of parameters for R . The images of these elements form a system of parameters in R/P . Then x_1, \dots, x_d is a regular sequence on R/P . But P has a finite filtration in which the factors are free (R/P) -modules, and so does R : one additional factor, R/P , is needed. Since x_1, \dots, x_d is a regular sequence on every factor of this filtration, by the Proposition near the bottom of the first page of the Lecture Notes from October 8, it is a regular sequence on R . Hence, R is Cohen-Macaulay. \square

Remark. It is also true that if M is a finitely generated module over an excellent ring R , then

$$\{P \in \text{Spec}(R) : M_P \text{ is Cohen-Macaulay}\}$$

is Zariski open in $\text{Spec}(R)$. This comes down to establishing property (2), and we may make the same initial reduction as in the ring case, killing a regular sequence in P on M whose image in PR_P is a maximal regular sequence on M_P . Therefore may assume that P is minimal in the support of M , and, after one further localization, that P is the only minimal prime in the support of M . We may assume as above that R/P is regular: again R

is Cohen-Macaulay suffices for the argument. We have $P^n M = 0$ for some n , and we may construct a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0$ such that every M_i/M_{i+1} is killed by P . We may then localize once more such that every M_i/M_{i+1} becomes (R/P) -free over the localization. The rest of the argument is the same as in the case where $M = R$. \square

There are several ways to prove that the type of a Cohen-Macaulay module cannot increase when one localizes. In particular, a Gorenstein ring remains Gorenstein when one localizes. One way is to make use of canonical modules. Here, we give a proof that is, in some sense, more elementary. Part of the argument is left as an exercise.

Theorem. *Let M be a module over a local ring (R, \mathfrak{m}, K) , and let P be any prime of R . Then the type of M_P is at most the type of M .*

Proof. We shall reduce to the case where R is a complete local domain of dimension one, M is a finitely generated torsion-free module, and $P = (0)$, so that the type of M_P is its dimension as a vector space over $\text{frac}(R)$. We leave this case as an exercise: see Problem 5 of Problem Set #4.

Let S be the completion of R , and let Q be a minimal prime of PS , which will lie over P . The closed fiber of $R_P \rightarrow S_Q$ is 0-dimensional because Q is minimal over PS , and so $M_P \otimes_{R_P} S_Q$ is Cohen-Macaulay and its type is the product of the type of M_P and the type of S_Q/PS_Q . This shows that the type of M_P is at most the type of $M_P \otimes_{R_P} S_Q \cong M \otimes_R S_Q = (\widehat{M})_Q$. Since M and \widehat{M} have the same type, it will suffice to show that type can not increase under localization in the complete case.

Second, we can choose a saturated chain of primes joining P to Q , and successively localize at each in turn. Thus, we can also reduce to the case where $\dim(R/Q) = 1$. Third, we can choose a maximal regular sequence on M in Q , and replace M by its quotient by this sequence. Thus, there is no loss of generality in assuming the Q is minimal in the support of M . We may also replace R by $R/\text{Ann}_R M$ and so assume that M is faithful. Let $N = \text{Ann}_M Q$. Let x be a system of parameters for R : it only has one element. In particular, $x \notin Q$. Then x is not a zerodivisor on M , since M is Cohen-Macaulay, nor on N , since $N \subseteq M$. But x is also not a zerodivisor on M/N , for if $xu \in N$, then $xuQ = 0$, which implies that $uQ = 0$, and so $u \in N$. It follows that when we apply $(R/xR) \otimes_R -$ to the short exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$$

we get an exact sequence. Hence, $N/xN \rightarrow M/xM$ is injective, which shows that the type of N is at most the type of M . However, $N_Q \subseteq M_Q$ is evidently the socle in M_Q , and so the type of M_Q is the same as the type of N_Q . It follows that it suffices to show that the type of N_Q is at most the type of N . Here, N is a torsion-free module over R/Q , and so we have reduced to the case described in the first paragraph. \square
