Math 711: Lecture of November 7, 2007

Our current theory of test elements permits the extension of many results proved under other hypotheses, such as the condition that the ring under consideration be a homomorphic image of a Cohen-Macaulay ring, to the case of excellent local rings or, more generally, rings for which we have completely stable test elements (or completely stable big test elements, depending on whether the result being proved is for finitely generated modules or for arbitrary modules).

Here is one example:

Theorem (colon-capturing). *Let (R, m, K) be an excellent reduced equidimensional local ring of prime characteristic $p > 0$, and let $x_1, \ldots, x_{k+1}$ be part of a system of parameters. Let $I_k = (x_1, \ldots, x_k)R$. Then $I_k^* :_R x_{k+1} = I_k^*$. In particular, $I_k :_R x_{k+1} \subseteq I_k^*$.*

Proof. Note that R has a completely stable test element c. Suppose that $ux_{k+1} \in I_k$ but $u \in R - I_k$. This is also true when we pass to $\widehat{R}$, which is a homomorphic image of a regular ring and, hence, of a Cohen-Macaulay ring. Therefore, from the result on colon-capturing from p. 9 of the Lecture Notes from October 5, we have that $u \in (I_k\widehat{R})^*$, whence $cu^q \in (I\widehat{R})^{[q]} = I^{[q]}\widehat{R}$ for all q, and it follows that $cu^q \in I^{[q]}\widehat{R} \cap R = I^{[q]}$ for all q. Thus, $u \in I^*$.

Now suppose that $ux_{k+1} \in I_k^*$. Then $u^q x_{k+1}^q \in (I_k^*)^{[q]} \subseteq (I_k^{[q]})^*$ for all q, and so $cu^q x_{k+1}^q \in I_k^{[q]}$ for all q, and $cu^q \in I_k^{[q]} :_R x_{k+1}^q \subseteq (I_k^{[q]})^*$ by the result of the first paragraph applied to $x_1^q, \ldots, x_{k+1}^q$. Hence, $c^2u^q \in (I_k)^{[q]}$ for all q, so $u \in I_k^*$. The opposite conclusion is obvious. $\square$

A Noetherian ring is called *locally excellent* if its localization at every maximal ideal (equivalently, at every prime ideal) is excellent.

Corollary. *If R is weakly F-regular and locally excellent, then R is Cohen-Macaulay.*

Proof. Both weak F-regularity and the Cohen-Macaulay property are local on the maximal ideals of R. Hence, we may assume that R is local. Since weakly F-regular rings are normal, R is certainly equidimensional. Since colon-capturing holds for systems of parameters in R, the result is immediate. $\square$

We can also prove a global version of this Theorem above that is valid even in case the ring is not equidimensional. We need one additional fact.

Lemma. *Let $(R,\, m,\, K)$ be an excellent local ring and let I be an ideal of R that has height at least k modulo every minimal prime of R. Then $I\widehat{R}$ has height at least k modulo every minimal prime of $\widehat{R}$.*

Proof. If $\mathfrak{p}_i$ is a minimal prime of R, $\mathfrak{p}_i\widehat{R}$ is a radical ideal and is the intersection of certain minimal primes $\mathfrak{q}_{ij}$. The intersection of all $\mathfrak{q}_{ij}$ is the same as the intersection of the $\mathfrak{p}_i\widehat{R}$. Since finite intersection commutes with flat base change, this is 0. Thus, it will suffice to show that the height of I is at least k modulo every $\mathfrak{q}_{ij}$. To this end, we can replace R by $R/\mathfrak{p}_i$. Thus, it is enough to show the result when R is an excellent local domain. In this case, $\widehat{R}$ is reduced and equidimensional. Any prime Q of $\widehat{R}$ containing $I\widehat{R}$ lies over a prime P of R containing I. The height of Q is at least the height of Q_0 where Q_0 is a minimal prime of $P\widehat{R}$. Hence, it suffices to show that if Q is a minimal prime of $P\widehat{R}$ then height $(Q) =$ height (P). Since R and $\widehat{R}$ are equidimensional and catenary,

$$\text{height}\, P = \dim(R) - \dim(R/P) = \dim(\widehat{R}) - \dim(\widehat{R}/P\widehat{R}).$$

Since the completion of R/P is equidimensional,

$$\dim(\widehat{R}/P\widehat{R}) = \dim\big(\widehat{R}/Q\big).$$

Hence,

$$\text{height}\,(P) = \dim(\widehat{R}) - \dim\big(\widehat{R}/Q\big) = \text{height}\,(Q),$$

as required. □

Theorem (colon-capturing). *Let R be a reduced Noetherian ring of prime characteristic $p > 0$ that is locally excellent and has a completely stable test element c . This holds, for example, if R is reduced and essentially of finite type over an excellent semilocal ring. Let $x_1,\, \dots,\, x_{k+1}$ be elements of R. Let I_t denote the ideal $(x_1,\, \dots,\, x_t)R$, $0 \le t \le k+1$. Suppose that the image of the ideal I_k has height k modulo every minimal prime of R, and that the image of the ideal $I_{k+1}R$ has height $k+1$ modulo every minimal prime of R. Then $I_k^* :_R x_{k+1} = I_k^*$.*

Proof. We first prove that $I_k :_R x_{k+1} \subseteq I_k^*$. The stronger conclusion then follows exactly as in the Theorem above because c is a test element.

If $x_{k+1}u \in I_k$ but $u \notin I_k^*$, we can choose q so that $cu^q \notin I_k^{[q]}$. This is preserved when we localize at a maximal ideal in the support of $(I_k^{[q]} + cu^q)/I_k^{[q]}$. We have therefore reduced to the case of an excellent local ring R_m. By the Lemma above, the hypotheses are preserved after completion, and we still have $cu^q \notin I_k^{[q]}S = (I_kS)^{[q]}$, where S is the completion of R_m. Since c is a test element in S, we have that $u \notin (I_kS)^*$. Since S is a homomorphic image of a Cohen-Macaulay ring, this contradicts the Theorem on colon-capturing from p. 9 of the Lecture Notes from October 5. □

We next want to use the theory of test elements to prove results on persistence of tight closure.

Persistence

Let $\mathcal{R}$ (respectively, $\mathcal{R}_{\text{big}}$) denote the class of Noetherian rings S such that for every domain $R = S/P$, the normalization R' of R is module-finite over R and has the following two properties:

(1) The singular locus in R' is closed.

(2) For every element $c \in R' - \{0\}$ such that R'_c is regular, c has a power that is a test element (respectively, a big test element) in R'.

Of course, $\mathcal{R}_{\text{big}} \subseteq \mathcal{R}$, and $\mathcal{R}_{\text{big}}$ includes both the class of F-finite rings and the class of rings essentially of finite type over an excellent semilocal ring.

Theorem (persistence of tight closure). *Let R be in $\mathcal{R}$ (respectively,in $\mathcal{R}_{\text{big}}$). Let $R \to S$ be a homomorphism of Noetherian rings and suppose that $N \subseteq M$ are finitely generated (respectively, arbitrary) R-modules. Let $u \in N^*_M$. Then $1 \otimes u \in \langle S \otimes_R N \rangle^*_{S \otimes_R M}$.*

Proof. It suffices to prove the result after passing to $S/\mathfrak{q}_j$ as $\mathfrak{q}_j$ runs through the minimal primes of S. Therefore, we may assume that S is a domain. Let P denote the kernel of $R \to S$. Then P contains a minimal prime $\mathfrak{p}$ of R. Tight closure persists when we kill $\mathfrak{p}$ because the element c used in the tight closure test is not in $\mathfrak{p}$. Hence, we may make a base change to $R/\mathfrak{p}$, and so we may assume that $R \to S$ is a map of domains with kernel P. It suffices to prove that tight closure is preserved when we pass from R to R/P, since the injective map of domains $R/P \hookrightarrow S$ always preserves tight closure. Henceforth we may assume that S has the form R/P.

Choose a saturated chain of primes

$$(0) = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_h = P.$$

Then it suffices to show that tight closure persists as we make successive base changes to R/P_1, then to R/P_2, and so forth, until we reach $R/P_h = R/P$. Therefore, we need only prove the result when $S = R/P$ and P has height one.

Let R' be the normalization of R and let Q be a prime of R' lying over P. We have a commutative diagram

$$\begin{array}{ccc} R/P & \longrightarrow & R'/Q \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array}$$

where the horizontal arrows are module-finite extensions and the vertical arrows are quotient maps. Tight closure is preserved by the base change from R to R' because it is an inclusion of domains. If R' is regular, then the element is in the image of the submodule, and this is preserved when we pass to R'/Q. If not, because R' is normal, the defining ideal of the singular locus has depth at least two, and we can find a regular sequence b, c in R' such that R'_b and R'_c are both regular. Then b and c are not both in Q, and hence at least one of them has nonzero image in R/Q: say that c has nonzero image. For some s, c^s is a test element (respectively, big test element) in R, and it has nonzero image in R'/Q. It follows that the base change $R' \to R'/Q$ preserves tight closure, and, hence, so does the composite base change $R \to R'/Q$.

Now suppose that the base change $R \to R/P$ fails to preserve tight closure. By the argument above, the further base change $R/P \hookrightarrow R'/Q$ restores the image of the element to the tight closure. This contradicts the fourth problem in Problem Set #4. Hence, $R \to R/P$ preserves tight closure. $\square$

Corolllary. *Let $R \to S$ be a homomorphism of Noetherian rings such that S has a completely stable (respectively, completely stable big) test element c and suppose that $N \subseteq M$ are finitely generated (respectively, arbitrary) R-modules. Let $u \in N^*_M$. Then $1 \otimes u \in \langle S \otimes_R N\rangle^*_{S\otimes_R M}$.*

Proof. Suppose that we have a counterexample. Then for some q, $c(1\otimes u)^q \notin \langle S \otimes_R N\rangle^{[q]}$. This continues to be the case after localization at a suitable maximal Q ideal of S, and after completing the local ring S_Q. Hence, we obtain a counterexample such that $(S, \mathfrak{n}, L)$ is a complete local ring. Let m be the contraction of $\mathfrak{n}$ to R and let let R_1 be the completion of R_m. Then the initial instance of tight closure is preserved by the base change from R to R_1 but not by the base change $R_1 \to S$. This is a contradiction, since R_1 is in $\mathcal{R}_{\text{big}}$. $\square$

Although the theory of test elements that we have developed thus far is reasonably satisfactory for theoretical purposes, it is useful to have theorems that assert that specific elements of the ring are test elements: not only that some unknown power is a test element.

For example, the following result is very useful.

Theorem. *Let R be a geometrically reduced, equidmensional algebra finitely generated over a field K of prime characteristic $p > 0$. Then the elements of the Jacobian ideal $\mathcal{J}(R/K)$ that are in R° are completely stable big test elements for R.*

It will be a while before we can prove this. We shall discuss the definition and properties of the Jacobian ideal in detail later. For the moment, we make only two comments. First, the Jacobian ideal defines the geometrically regular locus in R. Second, if $R = K[x_1, \ldots, x_n]/(f)$ is a hypersurface, then $\mathcal{J}(R/K)$ is simply the ideal generated by the images of the partial derivatives $\partial f/\partial x_i$ in R.

For example, suppose that K has characteristic $p > 0$ with $p \neq 3$. The Theorem above tells us that if $R = K[x, y, z]/(x^3+y^3+z^3)$ then the elements x^2, y^2, and z^2 are completely stable big test elements (3 is iinvertible in K). This is not the best possible result: the test ideal turns out to be all of $m = (x, y, z)$. But it gives a good starting point for computing $\tau(R)$.

The situation is essentially the same in the local case, where we study R_m instead. In this case, once we know that x^2, y^2 are test elements, we can calculate the test ideal as $(x^2, y^2) :_R (x^2, y^2)^*$, since this ring is Gorenstein and we may apply problem 4 of Problem Set #3. The socle generator modulo $I = (x^2, y^2)$ turns out to be xyz^2, and the problem of showing that the test ideal is mR_m reduces to showing that the ideal (x^2, y^2, xyz^2) is tightly closed in R_m. The main point here is that the Theorem above can sometimes be used to make the calculation of the test ideal completely down-to-earth.

The proof of the Theorem above involves several ingredients. One is the Lipman-Sathaye Jacobian Theorem, which we will state but not prove. The Theorem is proved in [J. Lipman and A. Sathaye, *Jacobian ideals and a theorem of Briançon-Skoda,* Michigan Math. J. **28** (1981) 199–222]. Moreover, a complete treatment of the Lipman-Sathaye argument is given in the Lecture Notes from Math 711, Fall 2006. See specifically, the Lectures of September 25, 27, and 29, as well as the Lectures of October 2, 4, 6, 9, 11, and 13.

Another ingredient is the Theorem stated just below. We shall say that an extension of a domain S of a field $\mathcal{K}$ is *étale* if S is a finite product of finite separable algebraic extension fields of $\mathcal{K}$. We shall say that an extension $A \to R$ of a domain A is *generically étale* if the generic fiber is étale, i.e., $\mathrm{frac}\,(A) \otimes_A R$ is a finite product of finite separable algebraic extension fields of $\mathrm{frac}\,(A)$.

Theorem. *Let R be a module-finite and generically étale extension of a regular ring A of prime characteristic $p > 0$. Let $r \in R^\circ$ be such that $cR^\infty \subseteq R[A^\infty]$. Then c is a completely stable big test element for R.*

We shall see that elements c as above exist, and that the Lipman-Sathaye Theorem can be used to find specific elements like this. The reason that it is very helpful that $cR^\infty \subseteq R[A^\infty]$ is that it turns out that $R[A^\infty] \cong R \otimes_A A^\infty$ is faithfully flat over R, since A^∞ is faithfully flat over A. This makes it far easier to work with $R[A^\infty]$ than it is to work with R^∞, and multiplication by c can be used to "correct" the error in replacing R^∞ by $R[A^\infty]$.

We begin by proving the following preliminary result.

Lemma. *Let (A, m_A, K) be a normal local domain and let R be a module-finite extension domain of A that is generically étale over A. Let* ord *be a $\mathbb{Z}$-valued valuation on A that is nonnegative on A and positive on m_A. Then* ord *extends uniquely to A^∞ by letting* $\mathrm{ord}\,(a^{1/q}) = (1/q)\mathrm{ord}\,(a)$ *for all $a \in A - \{0\}$. The extended valuation takes values in* $\mathbb{Z}[1/p]$.

Let $\mathfrak{m}$ be a proper ideal of R. Let d be the torsion-free rank of R as an A-module. Let $u \in \mathfrak{m}R[A^\infty] \cap A^\infty$. *Then* $\mathrm{ord}\,(u) \geq 1/d!$.

Proof. We know that $\mathrm{frac}\,(R)$ is separable of degree d over $\mathrm{frac}\,(A) = \mathcal{K}$. Let θ be a primitive element. The splitting field $\mathcal{L}$ of the minimal polynomial f of θ is generated by the roots of f, and is Galois over $\mathcal{K}$, with a Galois group that is a subgroup of the permutations on d, the roots of f. We may replace R by the possibly larger ring which is the integral closure of A in $\mathcal{L}$. Hence, we may assume that the extension of fraction fields is Galois with Galois group G, and $|G| = D \leq d!$. The action of G on R extends to $R[A^\infty]$, fixing A^∞. Let $x_1, \ldots, x_n \in R$ generate $\mathfrak{m}$. We have an equation

$$u = \sum_{i=1}^{n} a_i x_i$$

where the $a_i \in A^\infty$. The action of G then yields $|G|$ such equations:

$$(*_g) \quad u = \sum_{i=1}^{n} a_i g(x_i)$$

one for each element $g \in G$. Let $\underline{\nu} = (\nu_1, \ldots, \nu_n)$ run through n-tuples in $\mathbb{N}^n$ such that $\nu_1 + \cdots + \nu_n = D$, and let $\underline{\mathcal{P}}$ run through ordered partitions $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ of G into n sets. we use the notation $|\underline{\mathcal{P}}|$ for

$$(|\mathcal{P}_1|, \ldots, |\mathcal{P}_n|) \in \mathbb{N}^n.$$

Let

$$y_\nu = \sum_{|\underline{\mathcal{P}}|=\underline{\nu}} \left(\prod_{g \in \mathcal{P}_j} g(x_j) \right).$$

Then multiplying the elements $(*_g)$ together yields

$$u^D = \sum_{\underline{\nu}} a_1^{\nu_1} \cdots a_n^{\nu_n} y_{\underline{\nu}}.$$

Each $y_{\underline{\nu}}$ is invariant under the action of G, and is therefore in $R^G = A$. Since each $y_{\underline{\nu}}$ is a sum of products involving at least one of the x_i, each $y_j \in \mathfrak{m} \cap A \subseteq m_A$, and every $\operatorname{ord}(y_{\underline{\nu}}) \geq 1$. Hence,

$$D \operatorname{ord}(u) = \operatorname{ord}(u^D) \geq \min_\nu \operatorname{ord}(a_1^{\nu_1} \cdots a_n^{\nu_n} y_\nu) \geq \min_\nu \operatorname{ord}(y_\nu) \geq 1,$$

and we have that

$$\operatorname{ord}(u) \geq \frac{1}{D} \geq \frac{1}{d!},$$

as required. $\square$