Math 711: Lecture of November 9, 2007

We note the following fact from field theory:

Proposition. *Let $\mathcal{K}$ be a field of prime characteristic $p > 0$, let $\mathcal{L}$ be a separable algebraic extension of $\mathcal{K}$, and let $\mathcal{F}$ be a purely inseparable algebraic extension of $\mathcal{K}$. Then the map $\mathcal{F} \otimes_{\mathcal{K}} \mathcal{L}$ to the compositum $\mathcal{L}[\mathcal{F}]$ (which may be formed within a perfect closure or algebraic closure of $\mathcal{L}$) such that $a \otimes b \mapsto ab$ is an isomorphism.*

Proof. The map is certainly onto. It suffices to show that $\mathcal{F} \otimes_{\mathcal{K}} \mathcal{L}$ is a field: every element has a qth power in $\mathcal{L}$, and so if the ring is reduced it must be a field, and the injectivity of the map follows. $\mathcal{L}$ is a direct limit of finite separable algebraic extensions of $\mathcal{K}$, and so there is no loss of generality in assuming that the $\mathcal{L}$ is fnite over $\mathcal{K}$. The result now follows from the second Corollary on p. 4 of the Lecture Notes from September 19, or the argument given at the bottom of p. 6 of the Lecture Notes from October 19. $\square$

This fact is referred to as the *linear disjointness* of separable and purely inseparable field extensions.

In consequence:

Corollary. *Let $\mathcal{L}$ be a separable algebraic extension of $\mathcal{K}$, a field of prime characteristic $p > 0$. Then for every $q = p^e$, $\mathcal{L}[\mathcal{K}^{1/q}] = \mathcal{L}^{1/q}$.*

Proof. We need to show that every element of $\mathcal{L}$ has a qth root in $\mathcal{L}[\mathcal{K}^{1/q}]$. Since $\mathcal{L}$ is a directed union of finite separable algebraic extensions of $\mathcal{K}$, it suffices to prove the result when $\mathcal{L}$ is a finite separable algebraic field extension of $\mathcal{K}$. Let $[\mathcal{L} : \mathcal{K}] = d$. The field extension $\mathcal{K}^{1/q} \subseteq \mathcal{L}^{1/q}$ is isomorphic with the field extension $\mathcal{K} \subseteq \mathcal{L}$. Consequently, $[\mathcal{L}^{1/q} : \mathcal{K}^{1/q}] = d$ also. Since $\mathcal{K}^{1/q} \subseteq \mathcal{L}[\mathcal{K}^{1/q}] \subseteq \mathcal{L}^{1/q}$, to complete the proof it suffices to show that $[\mathcal{L}[\mathcal{K}^{1/q}] : \mathcal{K}^{1/q}] = d$ as well. But $\mathcal{L}[\mathcal{K}^{1/q}] \cong \mathcal{K}^{1/q} \otimes_{\mathcal{K}} \mathcal{L}$, and so its dimension as a $\mathcal{K}^{1/q}$-vector space is the same as the dimension of $\mathcal{L}$ as a $\mathcal{K}$-vector space, which is d. $\square$

We next prove:

Proposition. *Let R be module-finite, torsion-free, and generically étale over a regular domain A of prime characteristic $p > 0$.*

(a) *R is reduced.*

(b) *For every q, then map $A^{1/q} \otimes_A R \to R[A^{1/q}]$ is an isomorphism. Likewise, $A^{\infty} \otimes_A R \to R[A^{\infty}]$ is an isomorphism.*

(c) *For every q, $R[A^{1/q}]$ is faithfully flat over R. Moreover, $R[A^{\infty}]$ is faithfully flat over R.*

Proof. Let $\mathcal{K} = \mathrm{frac}\,(A)$. Then $\mathcal{K} \otimes_A R = \prod_{i=1}^h \mathcal{L}_i$, where every $\mathcal{L}_i$ is a finite separable algebraic extension of $\mathcal{K}$.

(a) Since R is torsion-free as an A-module, $R \subseteq \mathcal{K} \otimes_A R = \prod_{i=1}^h \mathcal{L}_i$, from which the result follows.

(b) We have an obvious surjection $A^{1/q} \otimes_A R \twoheadrightarrow R[A^{1/q}]$. Since R is torsion-free over A, each nonzero element $a \in A$ is a nonzerodivisor on R. Since $A^{1/q}$ is A-flat, this remains true when we apply $A^{1/q} \otimes_A \_$. It follows that $A^{1/q} \otimes_A R$ is a torsion-free A-module. Hence, we need only check that the map is injective after applying $\mathcal{K} \otimes_A \_$: if there is a kernel, it will not be killed. The left hand side becomes $\mathcal{K}^{1/q} \otimes_{\mathcal{K}} \prod_{i=1}^h \mathcal{L}_i$ and the right hand side becomes $\prod_{i=1}^h \mathcal{L}_i[\mathcal{K}^{1/q}]$. The map is the product of the maps $\mathcal{K}^{1/q} \otimes \mathcal{L}_i \to \mathcal{L}_i[\mathcal{K}^{1/q}]$, each of which is an isomorphism by the Proposition at the top of p. 1.

(c) This is immediate from part (b), since $A^{1/q}$ is faithfully flat over A for every q: this is equivalent to the flatness of $F^e : A \to A$. Since A^∞ is the directed union of the $A^{1/q}$, it is likewise flat over A, and since it is purely inseparable over A, it is faithfully flat. Hence, $R[A^\infty]$ is faithfully flat over R as well, by part (b). $\square$

Theorem. *Let R be module-finite, torsion-free, and generically étale over a regular domain A of prime characteristic $p > 0$. Then there exist nonzero elements $c \in A$ such that $cR^{1/p} \subseteq R[A^{1/p}]$. For such an element c, we have that $c^2 R^{1/q} \subseteq R[A^{1/q}]$ for all q, and, hence, also that $c^2 R^\infty \subseteq R[A^\infty]$.*

Proof. Consider the inclusion $R[A^{1/p}] \subseteq R^{1/p}$, which is a module finite extension: even $A^{1/p} \subseteq R^{1/p}$ is module-finite, because it is isomorphic with $A \subseteq R$. If we apply $\mathcal{K} \otimes_A \_$, on the left hand side R becomes $\prod_{i=1}^h \mathcal{L}_i$ and $A^{1/p}$ becomes $\mathcal{K}^{1/p}$. Hence, the left hand side becomes

$$(\prod_{i=1}^h \mathcal{L}_i)[\mathcal{K}^{1/p}] \cong \prod_{i=1}^h \mathcal{L}_i[\mathcal{K}^{1/p}] = \prod_{i=1}^h \mathcal{L}_i^{1/p}$$

by the Corollary on p. 1, and the right hand side also becomes $\prod_{i=1}^h \mathcal{L}_i^{1/p}$. Hence, if we take a finite set of generators for $R^{1/p}$ as an $R[A^{1/p}]$-module, each generator is multiplied into $R[A^{1/p}]$ by an element $c_i \in A^\circ$. The product c of the c_i is the required element.

Now suppose $c \in A^\circ$ is such that $cR^\infty \subseteq R[^\infty]$. Let

$$c_e = c^{1+\frac{1}{p}+\cdots+\frac{1}{p^e}}$$

and note that

$$c_{e+1} = c c_e^{1/p}$$

for $e \geq 1$. We shall show by induction on e that $c_e R^{1/q} \subseteq R[A^{1/q}]$ for every $e \in \mathbb{N}$. Note that $c_0 = c$, and this base case is given. Now suppose that $c_e R^{1/q} \subseteq R[A^{1/q}]$. Taking p th roots, we have that $c_e^{1/p} R^{1/pq} \subseteq R^{1/p}[A^{1/pq}]$. We multiply both sides by c to obtain

$$c_{e+1} R^{1/pq} = c c_e^{1/p} R^{1/pq} \subseteq c R^{1/p}[A^{1/pq}] \subseteq R[A^{1/p}][A^{1/pq}] = R[A^{1/pq}],$$

as required.

Since

$$1+\frac{1}{p}+\cdots+\frac{1}{p^e}\le 1+\frac{1}{2}+\cdots+\frac{1}{2^e}<2,$$

c^2 is a multiple of c_e in $A^{1/q}$ and in $R[A^{1/q}]$, and the stated result follows. $\square$

Also note:

Lemma. *Let R be module-finite, torsion-free, and generically étale over a domain A.*

(a) *If $A \hookrightarrow B$ is flat, injective homomorphism of domains, then $B \otimes_A R$ is module-finite, torsion-free and generically étale over B.*
In particular, if $\mathcal{K} \to \mathcal{L}$ is a field extension and $\mathcal{F}$ is an étale extension of $\mathcal{K}$, then $\mathcal{L} \otimes_{\mathcal{K}} \mathcal{F}$ is an étale extension of $\mathcal{L}$.

(b) *With the same hypothesis as in the first assertion in part (a), if $c \in R$ is such that $cR^\infty \subseteq R[A^\infty]$, then $c(B \otimes_A R)^\infty \subseteq (B \otimes_A R)[B^\infty]$.*

(c) *If $\mathfrak{q}$ is a minimal prime of R, then $\mathfrak{q}$ does not meet A and $A \hookrightarrow R/\mathfrak{q}$ is again module-finite, torsion-free, and generically étale over A.*

Proof. (a) We prove the second statement first. Since $\mathcal{F}$ is a product of finite separable algebraic extensions of $\mathcal{K}$, we reduce at once to the case where $\mathcal{F}$ is a finite separable algebraic field extension of K, and then $\mathcal{F} \cong \mathcal{K}[x]/(g)$, where x is an indeterminate and g is an irreducible monic polynomial of positive degree over $\mathcal{K}$ whose roots in an algebraic closure of $\mathcal{K}$ are mutually distinct. Let $g = g_1 \cdots g_s$ be the factorization of g into monic irreducible polynomials over $\mathcal{L}$. These are mutually distinct, and any two generate the unit ideal. Hence, by the Chinese Remainder Theorem,

$$\mathcal{L} \otimes_{\mathcal{K}} \mathcal{F} \cong \mathcal{L}[x]/(g) \cong \prod_{j=1}^{s} \mathcal{L}[x]/(g_j),$$

and every $\mathcal{L}_j[x]/g_j)$ is a finite separable algebraic extension of $\mathcal{L}$.

It is obvious that $B \otimes_A R$ is module-finite over A. By the Lemma on the first page of the Lecture Notes from October 12, the fact that R is module-finite and torsion-free over A implies that we have an embedding of $R \hookrightarrow A^{\oplus h}$ for some h. Because B is A-flat, we have injection $B \otimes_A R \hookrightarrow B^{\oplus h}$, and so $B \otimes_A R$ is torsion-free over B. The fact that the condition of being generically étale is preserved is immediate from the result of the first paragraph above, with $\mathcal{K} = \mathrm{frac}\,(A)$, $\mathcal{L} = \mathrm{frac}\,(B)$, and $\mathcal{F} = \mathcal{K} \otimes_A R$.

(b) Every element of $B \otimes_A R)^{1/q} \cong B^{1/q} \otimes_{A^{1/q}} R^{1/q}$ is a sum of elements of the form $b^{1/q} \otimes r^{1/q}$, while $c(b^{1/q} \otimes r^{1/q}) = b^{1/q} \otimes cr^{1/q}$. Since $cr^{1/q} \in R[A^\infty]$, it follows that $c(B \otimes_A R)^{1/q} \subseteq (B \otimes_A R)[A^\infty][B^{1/q}] \subseteq (B \otimes_A R)[B^\infty]$.

(c) $\mathfrak{q}$ cannot meet A° because R is torsion-free over A. Thus, $\mathfrak{q}$ corresponds to one of the primes of the generic fiber $\mathcal{K} \otimes_A R$, which is a product of finite algebraic separable field extensions of $\mathcal{K}$. It follows that $\mathcal{K} \otimes_A (R/\mathfrak{p})$ is one of these finite algebraic separable field extensions. $\square$

We can now prove the result on test elements, which we state again.

Theorem. *Let R be module-finite, torsion-free, and generically étale over a regular domain A. Let $c \in R^\circ$ be any element such that $cR^\infty \subseteq R[A^\infty]$. Then c is a completely stable big test element for R.*

Proof. Let Q be any prime ideal of R, and let P be its contraction to A. Then $A_P \to R_P$ satisfies the same hypothesis by parts (a) and (b) of the Lemma on p. 3, and so we may assume that A is local and that Q is a maximal ideal of R. We may now apply $B \otimes_A \_$, where $B = \widehat{A}$. By the same Lemma, the hypotheses are preserved. B becomes a product of complete local rings, one of which is the completion of R_Q. The hypotheses hold for each factor, and so we may assume without loss of generality that $(A,\, m,\, K) \to (R,\, \mathrm{frac}\, m,\, L)$ is a local map of complete local rings as well. Now suppose that $H \subseteq G$ are R-modules and $u \in H^*_G$. We may assume, as usual, that G is free. (This is not necessary, but may help to make the argument more transparent.) We are not assuming, however, that G or H is finitely generated.

We know that $u \in H^*_G$. Hence, there is an element r of R°, such that $ru^{q'} \in H^{[q']}$ for all $q' \gg 0$. Since $r \in R^\circ$, $\dim(R/rR) < \dim(R)$. Since R/rR is a module finite extension of $A/(rR \cap A)$, we must have that $\dim(A/(rR \cap A) < \dim(A)$, and it follows that $rR \cap A \neq (0)$, i.e., that r has a nonzero multiple $a \in A$. Then $au^{q'} \in H^{[q']}$ for all $q' \gg 0$. We may take q'th roots and obtain $a^{1/q'} \in R^{1/q'}H$ for all $q' \gg 0$. We are using the notation $R^{1/q'}H$ for the expansion of H to $R^{1/q'} \otimes_R G$, i.e., for the image of $R^{1/q'} \otimes H$ to $R^{1/q'} \otimes_R G$.

Let ord denote any valuation on A with values in $\mathbb{Z}$ that is nonnegative on A and positive on m. Then ord extends uniquely to a valuattion on A^∞ with values in $\mathbb{Z}[1/p]$ such that $\mathrm{ord}\,(b^{1/q'}) = (1/q')\mathrm{ord}\,(b)$ for all $b \in A^\circ$ and q'.

To complete the argument, we shall prove the following:

> (#) Suppose that $\{\delta_n\}_n$ is asequence of elements of $A^\infty - \{0\}$ such that $\delta_n u \in R^\infty H$ for all n and $\mathrm{ord}\,(\delta_n) \to 0$ as $n \to \infty$. Then $cu^q \in H^{[q]}$ for all q, and so $u \in H^*_G$.

This not only proves that c is a big test element, it also gives a new characterization of tight closure which is stated as a Corollary of this proof in the sequel.

Moreover, we will have proved that c is a completely stable big test element in every completed local ring of R, and so it will follow that c is a completely stable big test element for R.

If the statement (#) is false, fix q such that $cu^q \notin H^{[q]}$. For every n, we have

$$\delta_n u \in R^\infty H$$

and, hence,

$$\delta_n^q u^q \in R^\infty H^{[q]}.$$

Let $S = R[A^\infty]$, which we know is flat over R. We multiply by $c \in R$ to obtain $\delta_n^q(cu^q) \in SH^{[q]}$, i.e., that

$$\delta_n^q \in SH^{[q]} :_S cu^q = (H^{[q]} :_R cu^q)S,$$

by the second statement in part (a) of the Lemma on p. 1 of the Lecture Notes from October 29 and the flatness of S over R. Since $cu^q \notin H^{[q]}$, $H^{[q]} :_R cu^q$ is a proper ideal J of R, amd so $\delta_n^q \in JR[A^\infty]$ for all n. Now J is contained in some maximal ideal $\mathcal{M}$ of R. $\mathcal{M}$ contains a minimal prime $\mathfrak{q}$ of R. By part (c) of the Lemma on p. 3, A injects into $\overline{R}$, where $\overline{R}$ is module-finite domain extension of A generically étale over A, and $\mathfrak{m} = J\overline{R}$ is a proper ideal of $\overline{R}$. We can map $R[A^\infty]$ onto $\overline{R}[A^\infty]$ Then for all n, $\delta_n^q \in \mathfrak{m}\overline{R}[A^\infty] \cap A^\infty$. Let d denote the torsion-free rank of $\overline{R}$ over A. By the Lemma at the bottom of p. 5 of the Lecture Notes from November 7, we have that

$$q\,\mathrm{ord}\,(\delta_n) = \mathrm{ord}\,(\delta_n^q) \geq \frac{1}{d!}$$

for all n, and so

$$\mathrm{ord}\,\delta_n \geq \frac{1}{qd!}$$

for all n. Since q and d are both fixed and $n \to \infty$, this contradicts the assumption that $\mathrm{ord}\,(\delta_n) \to 0$ as $n \to \infty$. $\square$

Remark. The Theorem also holds when the regular ring A is not assumed to be a domain, if the hypothesis that R be torsion-free over A is taken to mean that every element of A° is a nonzero divisor on R, and the condition that R be generically étale over A is taken to mean that $R_\mathfrak{p}$ is étale over $A_\mathfrak{p}$ for every minimal prime $\mathfrak{p}$ of A. In this case, A is a finite product of regular domains, there is a corresponding product decomposition of R, and each factor of R is module-finite, torsion-free, and generically étale over the corresponding factor of A. The result follows at once from the results for the individual factors. $\square$

In the course of the proof of the Theorem we have demonstrated the following:

Corollary. *Let $(A,\, m,\, K)$ be a complete regular local ring, let R be module-finite, torsion-free and generically étale over A, and let* ord *by a $\mathbb{Z}$-valued valuation on A nonnegative on A and positive on m. Extend* ord *to a $\mathbb{Z}[1/p]$-valued valuation on A^∞. Let $N \subseteq M$ by R-modules and let $u \in M$. Then the following two conditions are equivalent:*

(1) $u \in N^*_M$.

(2) *There exists an infinite sequence of elements $\{\delta_n\}_n$ of $A^\infty - \{0\}$ such that* $\mathrm{ord}\,(\delta_n) \to 0$ *as $n \to \infty$ and for all n, $\delta_n \otimes u$ is in the image of $R^\infty \otimes_R N$ in $R^\infty \otimes_R M$.* $\square$

This result is surprising: in the standard definition of tight closure, the δ_n are all q th roots of a single element c. Here, they are permitted to be entirely unrelated. We shall soon prove a better result in this direction, in which the multipliers are allowed to be arbitrary elements of R^+ whose orders with respect to a valuation are approaching 0.