Math 711: Lecture of November 12, 2007

Before proceeding further with our treatment of test elements, we note the following consequence of the theory of approximately Gorenstein rings. We shall need similar splitting results in the proof of the generalization, stated in the first Theorem on p. 2, of the Corollary near the bottom of p. 5 of the Lecture Notes from November 9.

Theorem. *Let R be a weakly F-regular ring. Then R is a direct summand of every module-finite extension ring S. Moreover, if R is a complete local ring as well, R is a direct summand of R^+. In particular, these results hold when R is regular.*

Proof. Both weak F-regularity and the issue of whether $R \to S$ splits are local on the maximal ideals of R. Therefore, we may assume that (R, m, K) is local. Since R is approximately Gorenstein, there is a descending chain $\{I_t\}_t$ of m-primary irreducible ideals cofinal with the powers of m. By the splitting criterion in the Theorem at bottom of p. 4 and top of p. 5 of the Lecture Notes from October 24, R is a direct summand of S (or R^+ in case R is complete local) if and only if I_t is contracted for all t. In fact, $I_tS \cap R \subseteq I_t^*$ by the Theorem near the bottom of *p.* 1 of the Lecture Notes of October 12, and, by hypothesis, $I_t^* = I_t$. $\square$

It is an open question whether a locally excellent Noetherian domain R of prime characteristic $p > 0$ is weakly F-regular if and only if $(*)$ R is a direct summand of every module-finite extension ring. The issue is local on the maximal ideals of R, and reduces to the excellent local case. By the main result of [K. E. Smith, *Tight Closure of Parameter Ideals,* Inventiones Math. **115** (1994) 41–60], the tight closure of an ideal generated by part of a system of parameters is the same as its plus closure. From this result, it is easy to see that $(*)$ implies that R is F-rational. In the Gorenstein case, F-rational is equivalent to F-regular, so that the equivalence of the two conditions holds in the locally excellent Gorenstein case. We shall prove Smith's result that tight closure is the same as plus closure for parameter ideals. The argument depends on the use of local cohomology, and also utiliuzes general Néron desingularization. We shall also need the main result of [M. Hochster and C. Huneke, *Infinite integral extensions and big Cohen-Macaulay algebras,* Annals of Math. bf 135 (1992) 53–89], that if R is an excellent local domain, then R^+ is a big Cohen-Macaulay algebra over R. We shall prove this using a recent idea method of Huneke and Lyubeznik: cf. [C. Huneke and G. Lyubeznik, *Absolute integral closure in positive characteristic,* Advances in Math. **210** (2007) 498–504].

Our next immediate goal is to prove a strengthened version of the Corollary on p. 5 of the Lecture Notes from November 9. First note that if A is a regular local ring, we can choose a $\mathbb{Z}$-valued valuation ord that is nonnegative on A and positive on m. For example, if $a \neq 0$ we can let $\mathrm{ord}\,(a)$ be the largest integer k such that $a \in m^k$. We thus have an inclusion $A \subseteq V$ where V is a Noetherian discrete valuation ring. Now assume that A is complete, and complete V as well. That is, we have a local injection $A \hookrightarrow V$. We also have an injection $A^+ \hookrightarrow V^+$

For every module-finite extension domain R of A, where we think of R as a subring of A^+, we may form $V[R]$ within V^+. $V[R]$ is a complete local domain of dimension one that contains V. Its normalization, which we may form within V^+, is a complete local normal domain of dimension one, and is therefore a discrete valuation ring V_R. The generator of the maximal ideal of V is a unit times a power of the generator of the maximal ideal of V_R. Hence, ord extends to a valuation on R with values in the abelian group generated by $\dfrac{1}{h}$, where h is the order of the generator of the maximal ideal of V in V_R. Since A^+ is the union of all of these rings R, ord extends to a $\mathbb{Q}$-valued valuation on R^+ that is nonnegative on R^+ and postive on the maximal ideal of R^+.

If R is any complete local domain, we can represent R as a module-finite extension of a complete regular local ring A. Hence, we can choose a complete discrete valuation ring V_R and a local injection $R \to V_R$, and extend the corresponding $\mathbb{Z}$-valued valuation to a $\mathbb{Q}$-valued valuation that is nonnegative on R^+ and positive on the maximal ideal of R^+.

Theorem (valuation test for tight closure). *Let (R, m, K) be a complete local domain of prime characteristic $p > 0$ and let ord be a $\mathbb{Q}$-valued valuation on R^+ that is nonnegative on R^+ and positive on the maximal ideal of R^+. Let $N \subseteq M$ be arbitary R-modules and $u \in M$. Then the following two conditions are equivalent:*

(1) $u \in N^*_M$.

(2) *There exists a sequence $\{v_n\}$ of elements of $R^+ - \{0\}$ such that $\mathrm{ord}\,(v_n) \to 0$ as $n \to \infty$ and $v_n \otimes u$ is in the image of $R^+ \otimes_R N$ in $R^+ \otimes_R M$ for all n.*

We need several preliminary results in order to prove this.

The following generalization of colon-capturing can be further generalized in several ways. We only give a version sufficient for our needs here.

Theorem. *Let (R, m, K) be a reduced excellent local ring of prime characteristic $p > 0$. Let $x_1, \ldots, x_k \in m$ be part of a system of parameters modulo every minimal prime of R. Let $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{N}$, and assume that $a_i < b_i$ for all i. Then*

$$(x_1^{b_1}, \ldots, x_k^{b_k})^* :_R x_1^{a_1} \cdots x_k^{a_k} = (x_1^{b_1-a_1}, \ldots, x_k^{b_i-a_i})^*.$$

Proof. Let $d_i = b_i - a_i$. It is easy to see that each $x_i^{d_i}$ multiplies $x_1^{a_1} \cdots x_k^{a_k}$ into

$$(x_1^{b_1}, \ldots, x_k^{b_k}) \subseteq (x_1^{b_1}, \ldots, x_k^{b_k})^*,$$

since $d_i + a_i = b_i$ for every i. But if I is tightly closed, so is any ideal of the form $I :_R y$. (This is equivalent to the statement that if 0 is tightly closed in R/I, then it is also tightly closed in the smaller module $y(R/I) \cong R/(I :_R y)$.) Since

$$(x_1^{d_1}, \ldots, x_k^{d}) \subseteq (x_1^{b_1}, \ldots, x_k^{b_k})^* :_R x_1^{a_1} \cdots x_k^{a_k},$$

we also have

$$(x_1^{d_1}, \dots, x_k^{d_k})^* \subseteq (x_1^{b_1}, \dots, x_k^{b_k})^* :_R x_1^{a_1} \cdots x_k^{a_k}$$

Thus, it suffices to prove the opposite inclusion. By induction on the number of a_i that are not 0, we reduce at once to the case where only one of the a_i is not 0, because, quite generally,

$$I :_R (yz) = (I :_R y) :_R z.$$

By symmetry, we may assume that only $a_k \neq 0$. We write $x_k = x$, $a_k = a$, $b_k = b$ and $d_k = d$. Let $J = (x_1^{b_1}, \dots, x_{k-1}^{b_{k-1}})$. Suppose $x^a u \in (J + x^b R)^*$. Let $c \in R^\circ$ be a test element. Then $cx^{qa}u^q \in J^{[q]} + x^{qb}R$ for all $q \gg 0$, and for such q, we can write $cx^{qa}u^q = j_q + x^{qb}r_q$, where $j_q \in J^{[q]}$ and $r_q \in R$. Then $x^{qa}(cu^q - r_q x^{qd}) \in J^*$, and by the form of colon-capturing already established, we have that $cu^q - r_q x^{qd} \in (J^{[q]})^*$, and, hence,

$$c^2 u^q - cr_q x^{qd} \in J^{[q]}.$$

Consequently,

$$c^2 u^q \in J^{[q]} + x^{qd}R = (J + x^d R)^q$$

for all $q \gg 0$, and so $u \in (J + x^d R)^*$, as required. $\square$

Theorem. *Let (A, m, K) be a complete regular local ring of characteristic p and let* ord *be a $\mathbb{Q}$-valued valuation nonnegative on A^+, and positive on the maximal ideal of A^+. Let $v \in A^+ - \{0\}$ be an element such that* $\mathrm{ord}\,(v)$ *is strictly smaller than the order of any element of m (it suffices to check the generators of m). Then the map $A \to A^+$ such that $1 \mapsto v$ splits, i.e., there is A-linear map $\theta : A^+ \to A$ such that $\theta(v) = 1$.*

Proof. Let $x_1, \dots, x_n$ be minimal generators of m. Since A is complete and Gorenstein, it suffices, by the Theorem at the top of p. 3 of the Lecture Notes from October 24 to check that for all t,

$$x_1^t \cdots x_n^t v \notin (x_1^{t+1}, \dots, x_n^{t+1})A^+.$$

Suppose that

$$x_1^t \cdots x_n^t v = \sum_{i=1}^n s_i x_i^{t+!}.$$

Let $S = A[v, s_1, \dots, s_i]$. In S we have

$$v \in (x_1^{t+1}, \dots, x_n^{t+1})S :_S x_1^t \cdots x_k^t,$$

and so $v \in \big((x_1, \dots, x_n)S\big)^*$, by the preceding Theorem on colon-capturing. But then v is in the integral closure of $(x_1, \dots, x_n)S$, and this contradicts $\mathrm{ord}\,(v) < \min_i \mathrm{ord}\,(x_i)$. $\square$

Theorem. *Let $(R,\, m,\, K)$ be a complete local domain of prime characteristic $p>0$ and let $N \subseteq M$ be R-modules (not necessarily finitely generated). Let $u \in M$. Then the following conditions are equivalent:*

(1) $u \in N^*_M$.

(2) *There exist a fixed integer $s \in \mathbb{N}$ and arbitrarily large integers q such that $N^{[q]} :_R u^q$ meets $R - m^s$.*

(3) *There exist a fixed integer $s \in \mathbb{N}$ such that $N^{[q]} :_R u^q$ meets $R - m^s$ for all q.*

The following two conditons are also equivalent:

($1'$) $u \notin N^*_M$.

($2'$) *For all s, $N^{[q]} :_R u^q \subseteq m^s$ for all $q \gg 0$.*

Proof. Let $J_q = (N^{[q]})^* :_R u^q$, where $(N^{[q]})^* = (N^{[q]})^*_{\mathcal{F}^e(M)}$. Then the sequence of ideals J_q is descending. To see this, suppose that $r \in J_{pq}$. Then $ru^{pq} \in (N^{[pq]})^*$. Let c be a big test element for R. Then for all $q' \gg 0$,

$$cr^{q'}u^{pqq'} \in N^{[pqq']},$$

from which we have $c(ru^q)^{pq'} \in (N^{[q]})^{[pq']}$ since $r^{pq"}$ is a multiple of $r^{q'}$. This shows that $ru^q \in (N^{[q]})^*$ as well, and so $J_{pq} \subseteq J_q$ for every q.

Let $J = \bigcap_q J_q$. We shall show that whether $J \neq (0)$ or $J = (0)$ governs whether $u \notin N^*_M$ or $u \in N^*M$.

If $J \neq 0$ let $d \in J - \{0\}$. Then $du^q \in (N^{[q]})^*$ for all q, and then $(cd)u^q \in N^{[q]}$ for all q, and so $u \in N^*_M$.

If $J = (0)$, then by Chevalley's Lemma (see p. 6. of the Lecture Notes from October 24) we have that for all s the ideal $J :^{[q]} \subseteq m^s$ for al $q \gg 0$, and it follows as well that $N^{[q]} :_R u^q \subseteq m^s$ for all $q \gg 0$,

If $u \in J$ we have that $c \in N^{[q]} :_R u^q$ for all q. (3) obviously holds, since we can choose s such that $c \notin m^s$, and (3) $\Rightarrow$ (2). Now suppose (2) holds. If $u \notin N^*_M$, we have contradicted the result of the preceding paragraph. Hence, (1), (2), and (3) are equivalent.

The equivalence of ($1'$) and ($2'$) is the contrapositive of the equivalence of (1) and (2). $\square$