Math 711: Lecture of November 14, 2007

We continue to develop the preliminary results needed to prove the Theorem stated on p. 2 of the Lecture Notes from November 12. Only one more is needed.

Theorem. *Let $(R,\, m,\, K)$ be a complete local domain of prime characteristic $p > 0$, and let ord be a $\mathbb{Q}$-valued valuation nonnegative R^+ and positive on the maximal ideal of R^+. Then there exist an integer $s \geq 1$ and a positive rational number δ such that if $v \in R^+$ and $\mathrm{ord}\,(v) < \delta$, then there exists an R-linear map $\phi : R^+ \to R$ such that $\phi(v) \in R - m^s$.*

Proof. R is module-finite over a complete regular local ring A. Note that we may identify $R^+ = A^+$. Let δ be the minimum value of ord on a finite set of generators of the maximal m_A of A.

The module $\omega = \mathrm{Hom}_A(R,\, A)$ is a finitely generated A-module, but also has the structure of an R-module. Evidently, it is a finitely generated R-module. ω is torsion-free over A since its elements are functions with values in A. Since every nonzero element of R has a nonzero multiple in A, it is also torsion-free over R. Let $\mathcal{K} = \mathrm{frac}\,(A)$. Then $\mathcal{K} \otimes_A R = \mathcal{L}$ is the fraction field of R: let h be its degree, i.e., the torsion-free rank of R over A. Then

$$\mathcal{K} \otimes_A \omega \cong \mathrm{Hom}_{\mathcal{K}}(\mathcal{K} \otimes_A R,\, \mathcal{K}) \cong \mathrm{Hom}_{cK}(\mathcal{L}, \mathcal{K})$$

also has dimension d as a $\mathcal{K}$-vector space. Hence, as an $\mathcal{L}$-vector space, it must have dimension 1. Therefore, ω is a rank one torsion-free R-module, and so there exists an isomorphism $\omega \cong J \subseteq R$, where J is a nonzero ideal of R. Now $m_A R$ is primary to m, and so $m^k \subseteq m_A R$ for some k. We may apply the Artin-Rees Lemma to $J \subseteq R$ to conclude that

$$m^s \cap J = m^s R \cap J \subseteq m^k J$$

for s sufficiently large. Choose one such value of s. Then

$$m^s \cap J \subseteq m^k J \subseteq (m_A R) J = m_A J.$$

We shall prove that the desired conclusion holds for the values of δ and s that we have chosen. We may think of R^+ as A^+ and apply the Theorem on p. 3 of the Lecture Notes from November 12 to choose an A-linear map $\theta : R^+ \to A$ such that $\theta(v) = 1$. We then have an induced map

$$\mathrm{Hom}_A(R,\, R^+) \xrightarrow{\theta_*} \mathrm{Hom}_A(R,\, A)$$

which is R-linear, and hence a composite map

$$R^+ \xrightarrow{\mu} \mathrm{Hom}_A(R,\, R^+) \xrightarrow{\theta_*} \mathrm{Hom}_A(R,\, A) \xrightarrow{\cong} J \xrightarrow{\subseteq} R$$

where the first map μ is the map that takes $u \in R^+$ to the map $f_u : R \to R^+$ such that $f_u(r) = ur$ for all $r \in R$. Note that μ is R-linear using the R-module structure on $\mathrm{Hom}_A(R, R^+)$ that comes from R, since $r'u$ maps to $f_{r'u}$ and

$$f_{r'u}(r) = r'ur = u(r'r) = f_u(r'r) = (r'f_u)(r).$$

Call the composite map ϕ.

We shall prove that ϕ has the required property. First note that $(\theta_* \circ \mu)(v) \in \mathrm{Hom}_A(R, A)$ is very special: its value on 1 is $\theta(v \cdot 1) = \theta(v) = 1$. Thus, it is a splitting of the inclusion map $A \hookrightarrow R$. But this means that $(\theta_* \circ \mu)(v) \notin m_A \mathrm{Hom}(R, A)$, for maps in $m_A \mathrm{Hom}_A(R, A)$ can only take on values that are in m_A. It follows that $\phi(v) \notin m_A J \subseteq R$. Since $m^s \cap J \subseteq m_A J$, it also follows that $\phi(v) \notin m^s$, as required. $\square$

We are now ready to prove the first Theorem stated on p. 2 of the Lecture Notes from November 12.

Proof of the valuation test for tight closure. We may assume without loss of generality that $M = G$ is free, that $N = H \subseteq G$, and that $u \in G$. We fix a basis for G, and identify $\mathcal{F}^e(G) \cong G$. We identify H with its image $1 \otimes H \subseteq R^+ \otimes_R G$, and write R^+H for $\langle R^+ \otimes_R H \rangle$ which we may think of as all R^+-linear combinations of elements of H in $R^+ \otimes_R G$. We shall simply write wg for $w \otimes g$ when $w \in R^+$ and $g \in G$.

Suppose that we have $v_n u \in R^+H$ for all n and $\mathrm{ord}\,(v_n) \to 0$. Choose s and δ as in the preceding Theorem. Fix q, and choose n so large that $\mathrm{ord}\,(v_n) < \delta/q$, so that $\mathrm{ord}\,(v_n^q) < \delta$. Choose an R-linear map $\phi : R^+ \to R$ such that $\phi(v_n^q) \in R - m^s$. Then tensoring with G yields an R-linear map $R^+ \otimes_R G \to G$ such that $wg \mapsto \phi(w)g$ for every $w \in R^+$ and $g \in G$.

Since $v_n u \in R^+H$, we may apply $\mathcal{F}^e$ to obtain that

$$v_n^q u^q \in R^+ H^{[q]}.$$

We may now apply $\phi \otimes \mathbf{1}_G$ and conclude that

$$\phi(v_n^q) u^q \in H^{[q]},$$

where $\phi(v_n^q) \in R - m^s$. Hence, for every q, $H^{[q]} :_R u^q$ meets $R - m^s$. By the Theorem at the top of p. 4 of the Lecture Notes from November 12, $u \in H^*_G$, as required. $\square$

Capturing normalizations using discriminants

Consider $A \to R$ where A is a normal or possibly even regular Noetherian domain and R is module-finite, torsion-free, and generically étale over A. In this situation, we know that R is reduced. It has a normalization R' in its total quotient ring $\mathcal{T}$. Under these hypotheses, $\mathcal{T}$ may be identified with $\mathcal{K} \otimes_A R$, which is a finite product of finite separable algebraic field extensions $\mathcal{L}_1 \times \cdots \times \mathcal{L}_h$ of $\mathcal{K}$. R' is the product of the normalizations R'_i of the domains $R_i = R/\mathfrak{p}_i$ obtained by killing a minimal prime $\mathfrak{p}_i$ of R. There is one minimal prime for each $\mathcal{L}_i$, namely, the kernel of the composite homomorphism

$$R \to \mathcal{K} \otimes_A R \cong \prod_{i=1}^{h} \mathcal{L}_i \twoheadrightarrow \mathcal{L}_i,$$

and $\mathcal{L}_i$ is the fraction field of the domain R_i.

We want to develop methods of finding elements $c \in R^\circ$ that "capture" the normalization R' in the sense that $cR' \subseteq R$. Moreover, we want a construction such that c continues to have this property after a flat injective base change $A \hookrightarrow B$ of regular domains. We shall see that whenever we have such an element $c \in R^\circ$, it is a big completely stable test element.

There are two methods for constructing such elements c. One is to take c to be a discriminant for R over A: we explain this idea in the immediate sequel. The other is to use the Lipman-Sathaye Jacobian Theorem.

The use of discriminants is much more elementary, and we explore this method first.

Discussion: discriminants. Let A be a normal Noetherian domain and let R be a module-finite torsion-free generically étale extension of A. Let $\mathcal{K} = \mathrm{frac}\,(A)$ and

$$\mathcal{T} = \mathcal{K} \otimes_A R \cong \prod_{i=1}^{h} \mathcal{L}_i,$$

where the $\mathcal{L}_i$ are finite separable algebraic field extensions of $\mathcal{K}$. We first note that we have a trace map $\mathrm{Trace}_{\mathcal{T}/\mathcal{K}} : \mathcal{T} \to \mathcal{K}$ that is $\mathcal{K}$ linear. Such a map is defined whenever $\mathcal{T}$ is a $\mathcal{K}$-algebra that is finite-dimensional as as $\mathcal{K}$-vector space. The *trace* of $\lambda \in \mathcal{T}$ is defined to be the trace of the linear transformation $f_\lambda : \mathcal{T} \to \mathcal{T}$ that sends $\alpha \mapsto \lambda\alpha$. To calculate the trace, one takes a basis for $\mathcal{T}$ over $\mathcal{K}$, finds the matrix of multiplication by λ, and then takes the sum of the diagonal entries of the matrix. The trace is independent of how one chooses the $\mathcal{K}$-basis for $\mathcal{T}$. We summarize thes properties as follows.

Proposition. *Let $\mathcal{T}$ be a $\mathcal{K}$-algebra that is finite-dimensional as a $\mathcal{K}$-vector space. Then* $\mathrm{Trace}_{\mathcal{T}/\mathcal{K}} : \mathcal{T} \to \mathcal{K}$ *is a $\mathcal{K}$-linear map.*

If $\mathcal{K}'$ is any field extension of $\mathcal{K}$ and $\mathcal{T}' = \mathcal{K}' \otimes_{\mathcal{K}} \mathcal{T}$, then

$$\mathrm{Trace}_{\mathcal{T}'/\mathcal{K}'} = \mathbf{1}_{\mathcal{K}'} \otimes_{\mathcal{K}} \mathrm{Trace}\mathcal{T}/\mathcal{K},$$

i.e., if $\beta \in \mathcal{K}'$ and $\lambda \in \mathcal{T}$, then

$$\mathrm{Trace}_{\mathcal{T}'/\mathcal{K}'}(\beta \otimes \lambda) = \beta\,\mathrm{Trace}_{\mathcal{T}/\mathcal{K}}(\lambda).$$

Proof. The previous discussion established the first statement, while the second statement is a consequence of the following two observations. First, if $\theta_1, \dots, \theta_n$ is a basis for $\mathcal{T}$ over $\mathcal{K}$, then the images $1\otimes\theta_1, \dots, 1\otimes\theta_n$ of these elements in $\mathcal{K}'\otimes_{\mathcal{K}}\mathcal{T}$ form a basis for $\mathcal{K}'\otimes_{\mathcal{K}}\mathcal{T}$ over $\mathcal{K}'$. Second, If the matrix for multiplication by λ with respect to the basis $\theta_1, \dots, \theta_n$ is $\mathcal{M}$, the matrix for multiplication by $\beta\otimes\lambda$ with respect to the basis $1\otimes\theta_1, \dots, 1\otimes\theta_n$ is $\beta\mathcal{M}$, from which the stated result is immediate. $\square$

Because A is normal, the restriction of the trace map to R takes values in A, not just in $\mathcal{K}$. One may argue as follows: a normal Noetherian domain is the intersecftion of the discrete valuation domains V of the form $A_P \subseteq \mathcal{K}$, where P is a height one prime of A. Thus, we may make a base change from A to such a ring V, and it suffices to prove the result when $A = V$ is a Noetherian discrete valuation ring. In this case, because R is torsion-free, it is free, and we may choose a free basis for R over V. This will also be a basis for $\mathcal{L}$ over $\mathcal{K}$. If we compute trace using this basis, all entries of the matrix are in V, and so it is obvious that the trace is in V.

We note that when $\mathcal{T} = \prod_{i=1}^{h}\mathcal{L}_i$ is a finite product of finite separable algebraic field extensions, the map $B : \mathcal{T}\times\mathcal{T} \to \mathcal{K}$ that sends $(\lambda, \lambda') \mapsto \mathrm{Trace}_{\mathcal{T}/\mathcal{K}}(\lambda\lambda')$ is a nondegenerate symmetric bilinear form. This is a well known characterization of separability of finite algebraic field extensions if there is only one $\mathcal{L}_i$. See the Lecture Notes from December 3 from Math 614, Fall 2003, pp. 4–6. In the general case, choose a basis for each $\mathcal{L}_i$ and use the union as a basis for $\mathcal{T}$. It follows at once that

$$\mathrm{Trace}_{\mathcal{T}/\mathcal{K}}(\lambda_1, \dots, \lambda_h) = \sum_{i=1}^{h}\mathrm{Trace}_{\mathcal{T}_i/\mathcal{K}}(\lambda_i),$$

since the matrix of multiplication by $(\lambda_1, \dots, \lambda_h)$ is the direct sum of the matrices of multiplciation by the individual λ_i. To show non-degeneracy, consider a nonzero element $\lambda = (\lambda_1, \dots, \lambda_h)$. Then some λ_i is not 0: by renumbering, we may assume that $\lambda_1 \neq 0$. Since $\mathrm{Trace}_{\mathcal{L}_1/\mathcal{K}}$ yields a nondegenerate bilinear form on $\mathcal{L}_1$, we can choose $\lambda_1' \in \mathcal{L}_1$ such that

$$\mathrm{Trace}_{\mathcal{L}_1/\mathcal{K}}(\lambda_1'\lambda_1) \neq 0.$$

Let $\lambda' \in \mathcal{L}$ be the element $(\lambda_1', 0, \dots, 0)$. Then

$$\mathrm{Trace}_{\mathcal{T}/\mathcal{K}}(\lambda'\lambda) = \mathrm{Trace}_{\mathcal{L}_1/\mathcal{K}}(\lambda_1'\lambda_1) \neq 0.$$

By a *discriminant* for R over A we mean an element of A obtained as follows: choose $\underline{\theta} = \theta_1, \dots, \theta_n \in R$ whose images in $\mathcal{L}$ are a basis for $\mathcal{T}$ over $\mathcal{K}$, and let

$$D = D_{\underline{\theta}} = \det\bigl(\mathrm{Trace}_{\mathcal{L}/\mathcal{K}}(\theta_i\theta_j)\bigr).$$

Since every $\theta_i\theta_j \in R$, the matrix has entries in A, and the determinant is in A. Since the corresponding bilinear form is nondegenerate, D is a nonzero element of A. The following result is very easy.

Proposition. *Let R be module-finite, torsion-free, and generically étale over a normal Noetherian domain A. Let $A \hookrightarrow B$ be a flat injective map of normal Noetherian domains.*

(a) *$B \to B \otimes_A R$ is module-finite, torsion-free, and generically étale.*

(b) *If the images of $\theta_1, \ldots, \theta_n \in R$ in $\mathcal{K} \otimes_A R$ are a $\mathcal{K}$ basis for $\mathcal{K} \otimes_A R$, then their images $\underline{\theta}' = 1 \otimes \theta_1, \ldots, 1 \otimes \theta_n$ in $B \otimes_A R$ are a basis for $\mathrm{frac}\,(B) \otimes_B (B \otimes_A R)$ over $\mathrm{frac}\,(B)$.*

(c) *The image of the discriminant $D_{\underline{\theta}}$ in B is the discriminant $D_{\underline{\theta}'}$ of the basis $\theta' = 1 \otimes \theta_1, \ldots, 1 \otimes \theta_n$ for the extension $B \to B \otimes_A R$.*

Proof. We have already proved part (a): this is part (a) of the Lemma on p. 3 of the Lecture Notes from November 9. Part (b) is obvious, and part (c) follows from part (b), the definition of the discriminant, and the second statement of the Proposition on p. 3. $\square$

The key property of discriminants for us is:

Theorem. *Let R be module-finite, torsion-free, and generically étale over a normal Noetherian doman A with fraction field $\mathcal{K}$. Let $\theta_1, \ldots, \theta_n \in R$ give a $\mathcal{K}$-vector space basis for $\mathcal{T} = \mathcal{K} \otimes_A R$. Let R' denote the normalization of R in its total quotient ring $\mathcal{A} \otimes_A R$. Then $D_{\underline{\theta}}$, the discriminant of $\theta_1, \ldots, \theta_n$, is a nonzero element of A such that $D_{\underline{\theta}} R' \subseteq R$. In consequence, R' is module-finite over R.*

Proof. Let $s \in R'$. Then $s \in \mathcal{K} \otimes R$, and so we can write s uniquely in the form

$$(\#) \quad s = \sum_{j=1}^{n} \alpha_j \theta_j,$$

where the $\alpha_i \in \mathcal{K}$. For every θ_i we have that

$$\theta_i s = \sum_{j=1}^{n} \alpha_i \theta_i \theta_j.$$

Let Trace denote $\mathrm{Trace}_{\mathcal{T}/\mathcal{K}}$, and apply the $\mathcal{K}$-linear operator Trace to both sides of the equation. Since $\theta_i s$ is integral over R, its trace a_i is in A. This gives n equations

$$a_i = \sum_{j=1}^{n} \alpha_j \mathrm{Trace}(\theta_i \theta_j),\ 1 \leq i \leq n.$$

Let $\mathcal{M}$ denote the matrix $\big(\mathrm{Trace}(\theta_i \theta_j)\big)$, which has entries in A. Then these equations give a matrix equation

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \mathcal{M} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Let $\mathrm{adj}(\mathcal{M})$ denote the classical adjoint of $\mathcal{M}$, the transpose of the matrix of cofactors. Then $\mathrm{adj}\,(\mathcal{M})\mathcal{M} = D\boldsymbol{I}_n$ where $D = \det(\mathcal{M}) = D_{\underline{\theta}}$, and $\boldsymbol{I}_n$ is the size n identity matrix. Note that $\mathrm{adj}(\mathcal{M})$ has entries in A. Then

$$\begin{pmatrix} D\alpha_1 \\ \vdots \\ D\alpha_n \end{pmatrix} = D\boldsymbol{I}_n \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \mathrm{adj}\,(\mathcal{M})\mathcal{M} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \mathrm{adj}\,(\mathcal{M}) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix},$$

and the rightmost column matrix clearly has entries in A. Hence, every $D\alpha_i \in A$, and it follows from the displayed formula (#) on the preceding page that $Ds \in R$. Since $s \in R'$ was arbitrary, we have proved that $DR' \subseteq R$, as requried.

The final statement follows because $R' \cong DR' \subseteq R$ as A-modules, and so R' is finitely generated over A and, hence, over R. $\square$