Math 711: Lecture of November 16, 2007

We next observe:

Lemma. *Let A be a regular Noetherian domain of prime characteristic $p > 0$, and let $A \to R$ be module-finite, torsion-free, and generically étale. Then for every q, $R^{1/q}$ is contained in the normalization of $R[A^{1/q}]$.*

Proof. Fix $q = p^e$, and let

$$S = R[A^{1/q]} \cong A^{1/q} \otimes_A R.$$

Since every element of $R^{1/q}$ has its q th power in $R \subseteq S$, the extension is integral. Let $\mathcal{K} = \mathrm{frac}\,(A)$. We can write

$$\mathcal{K} \otimes_A R = \prod_{i=1}^{h} \mathcal{L}_i$$

where every $\mathcal{L}_i$ is finite separable algebraic extension field $\mathrm{frac}\,(A) = \mathcal{K}$. Then

$$\mathcal{K} \otimes_A R^{1/q} = \mathcal{K}^{1/q} \otimes_{A^{1/q}} R^{1/q},$$

since $R^{1/q}$ contains $A^{1/q}$ and inverting every element of $A - \{0\}$ makes every element of $A^{1/q} - \{0\}$ invertible. Hence,

$$\mathcal{K} \otimes_A R^{1/q} \cong \prod_{i=1}^{h} \mathcal{L}_i^{1/q}.$$

This is the total quotient ring of $R^{1/q}$. On the other hand,

$$\mathcal{K} \otimes_A (A^{1/q} \otimes_A R) \cong \mathcal{K}^{1/q} \otimes_{\mathcal{K}} (\mathcal{K} \otimes_A R) \cong \mathcal{K}^{1/q} \otimes_{\mathcal{K}} \prod_{i=1}^{h} \mathcal{L}_i \cong \prod_{i=1}^{h} (\mathcal{K}^{1/q} \otimes_{\mathcal{K}} \mathcal{L}_i) \cong \prod_{i=1}^{h} \mathcal{L}_i^{1/q}$$

where the rightmost isomorphism follows from the Corollary on p. 1 of the Lecture Notes from November 9. Thus, $R^{1/q}$ is contained in the total quotient ring of $R[A^{1/q}]$. $\square$

We can now show that discriminants yield test elements.

Theorem. *Let A be a regular Noetherian domain of prime characteristic $p > 0$, and let $A \to R$ be module-finite, torsion-free, and generically étale. Let $\mathcal{K} = \mathrm{frac}\,(A)$, let $\underline{\theta} = \theta_1, \ldots, \theta_n$ be elements of R that form a basis for $\mathcal{T} = \mathcal{K} \otimes_{\mathcal{K}} R$, and let $D = D_{\underline{\theta}} \in A^{\circ}$ be the discriminant. Then D is a competely stable big test element for R.*

Moreover, if $A \to B$ is an injective flat homomorphism of regular domains, then the image of D in $B \otimes_A R$ is a completely stable big test element for $B \otimes_A R$.

Proof. It follows from the Lemma above and the Theorem on p. 5 of the Lecture Notes of November 14 that $DR^{1/q} \subseteq R[A^{1/q}]$ for all q, Hence, by the Theorem on p. 4 of the Lecture Notes of November 9, D is a completely stable big test element for R.

The final statement now follows because, by parts (a) and (b) of the Lemma on p. 3 of the Lecture Notes from November 9 and the Proposition at the top of p. 5 of the Lecture Notes from November 14, the hypotheses are preserved by the base change from A to B. $\square$

We next want to show that the Lipman-Sathaye Jacobian Theorem produces test elements in an entirely similar way: it also provides elements that "capture" the normalization of an extension of A and are stable under suitable base change.

Test elements using the Lipman-Sathaye Jacobian Theorem

Until further notice, A denotes a Noetherian domain with fraction field $\mathcal{K}$, and R denotes an algebra essentially of finite type over A such that R is torsion-free and *generically étale* over A, by which we mean that $\mathcal{T} = \mathcal{K} \otimes_R S$ is a finite product of finite separable algebraic field extensions of $\mathcal{K}$. Note that $\mathcal{T}$ may also be described as the total quotient ring of S. We shall denote by R' the integral closure of R in $\mathcal{T}$.

We shall write $\mathcal{J}_{R/A}$ for the *Jacobian ideal* of R over A. If R is a finitely generated A-algebra, so that we may think of R as

$$A[X_1, \dots, X_n]/(f_1, \dots, f_h),$$

then $\mathcal{J}_{R/A}$ is the ideal of R generated by the images of the size n minors of the Jacobian matrix $(\partial f_j/\partial x_i)$ under the surjection $A[X] \to R$. This turns out to be independent of the presentation, as we shall show below. Moreover, if $u \in R$, then $\mathcal{J}_{R_u/A} = \mathcal{J}_{R/A}R_u$. From this one sees that when R is essentially of finite type over A and one defines $\mathcal{J}_{S/R}$ by choosing a finitely generated subalgebra R_0 of R such that $R = W^{-1}R_0$ for some multiplicative system W of R_0, if one takes $\mathcal{J}_{R/A}$ to be $\mathcal{J}_{R_0/A}R$, then $\mathcal{J}_{R/A}$ is independent of the choices made. We shall consider the definition in greater detail later. We use Jacobian ideals to state the following result, which we shall use without proof here.

Theorem (Lipman-Sathaye Jacobian theorem). *Let A be regular domain with fraction field $\mathcal{K}$ and let R be an extension algebra essentially of finite type over A such that R is torsion-free and generically étale over A. Let $\mathcal{T} = \mathcal{K} \otimes_A R$ and let R' be the integral closure*[1] *of R in $\mathcal{L}$. If $\theta \in \mathcal{T}$ is such that $\theta \mathcal{J}_{R'/A} \subseteq R'$ then $\theta \mathcal{J}_{R/A} R' \subseteq R$.*

In particular, we may take $\theta = 1$. Thus, if $c \in \mathcal{J}_{R/A}$, then $cR' \subseteq R$.

In other words, elements of $\mathcal{J}_{R/A}$ "capture" the integral closure of R: as in the case of discriminants, this will enable us to prove the existence of completely stable big test elements.

[1] One can show that R' is module-finite over R.

In constructing test elements, we shall only use the Theorem in the case where R is module-finite over A as well. In this case, we already know from our treatment of discriminants that R' is module-finite over R. The result that R' is module-finite over R in the more general situation is considerably more difficult: see the Theorem at the bottom of p. 1 of the Lecture Notes from October 4 from Math 711, Fall 2006.

The Lipman-Sathaye Theorem is proved in [J. Lipman and A. Sathaye, *Jacobian ideals and a theorem of Briançon-Skoda,* Michigan Math. J. **28** (1981) 199–222], although it is assumed that R is a domain in that paper. The argument works without essential change in the generality stated here. See also [M. Hochster, *Presentation depth and the Lipman-Sathaye Jacobian theorem*, Homology, Homotopy and Applications (International Press, Cambridge, MA) **4** (2002) 295–314] for this and further generalizations. As mentioned earlier, the proof of the Jacobian Theorem is given in the Lecture Notes from Math 711, Fall 2006, especially the Lecture Notes from September 25, 27, and 29, as well as the Lectures of October 2, 4, 6, 9, 11, and 13.

We note that the hypotheses of the Lipman-Sathaye are stable under flat base change.

Proposition. *Let R be essentially of finite type, torsion-free, and generically étale over the regular domain A, and let $A \hookrightarrow B$ be a flat injective homomorphism to a regular domain B. Then $B \otimes_A R$ is essentially of finite type, torsion-free, and generically étale over B.*

Morever, $\mathcal{J}_{(B\otimes_A R)/B} = \mathcal{J}_{R/A}(B \otimes_A R)$.

Proof. The proof that $B \otimes_A R$ is essentially of finite type over B is completely straightforward. The proof that the extension remains generically étale is the same as in the Lemma on p. 3 of the Lecture Notes from Novervmber 9. The proof that $B \otimes_A R$ is torsion-free over B needs to be modified slightly. R is a directed union of finitely generated torsion-free A-modules N. Each such N is a submodule of a finitely generated free A-module. Hence, each $B \otimes_A N$ is a submodule of a finitely generated free B-module. Since $B \otimes_A R$ is the directed union of these, it is torsion-free over B.

The proof of the last statement reduces at once to the case where R is finitely generated over A. Suppose that

$$R = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m).$$

Then

$$B \otimes_A R \cong B[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

(the ideal in the denominator now an ideal of a larger ring, but it has the same generators). The result is now immediate form the definition. $\square$

Why the Jacobian ideal is well-defined

We next want to explain why the Jacobian ideal is well-defined. We assume first that R is finitely generated over A. Suppose that $R = A[x_1, \dots, x_n]/I$. To establish independence of the presentation we first show that the Jacobian ideal is independent of the choice of generators for the ideal I. Obviously, it can only increase as we use more generators. By enlarging the set of generators still further we may assume that the new generators are obtained from the orginal ones by operations of two kinds: multiplying one of the original generators by an element of the ring, or adding two of the original generators together. Let us denote by ∇f the column vector consisting of the partial derivatives of f with respect to the variables. Since $\nabla(gf) = g\nabla f + f\nabla g$ and the image of a generator f in R is 0, it follows that the image of $\nabla(gf)$ in R is the same as the image of $g\nabla f$ when $f \in I$. Therefore, the minors formed using $\nabla(gf)$ as a column are multiples of corresponding minors using ∇f instead, once we take images in R. Since $\nabla(f_1 + f_2) = \nabla f_1 + \nabla f_2$, minors formed using $\nabla(f_1 + f_2)$ as a column are sums of minors from the original matrix. Thus, independence from the choice of generators of I follows.

Now consider two different sets of generators for R over A. We may compare the Jacobian ideals obtained from each with that obtained from their union. This, it suffices to check that the Jacobian ideal does not change when we enlarge the set of generators $f_1, \dots, f_s$ of the algebra. By induction, it suffices to consider what happens when we increase the number of generators by one. If the new generator is $f = f_{s+1}$ then we may choose a polynomial $h \in A[X_1, \dots, X_s]$ such that $f = h(f_1, \dots, f_s)$, and if $g_1, \dots, g_h$ are generators of the original ideal then $g_1, \dots, g_h$, $X_{s+1} - h(X_1, \dots, X_s)$ give generators of the new ideal. Both dimensions of the Jacobian matrix increase by one: the original matrix is in the upper left corner, and the new bottom row is $(0\ 0\ \dots\ 0\ 1)$. The result is then immediate from

Lemma. *Consider an $h+1$ by $s+1$ matrix $\mathcal{M}$ over a ring R such that the last row is $(0\ 0\ \dots\ 0\ u)$, where u is a unit of R. Let $\mathcal{M}_0$ be the h by s matrix in the upper left corner of $\mathcal{M}$ obtained by omitting the last row and the last column. Then $I_s(\mathcal{M}_0) = I_{s+1}(\mathcal{M})$.*

Proof. If we expand a size $s+1$ minor with respect to its last column, we get an R-linear combination of size s minors of $\mathcal{M}_0$. Therefore, $I_{s+1}(\mathcal{M}) \subseteq I_s(\mathcal{M}_0)$. To prove the other inclusion, consider any s by s submatrix Δ_0 of $\mathcal{M}_0$. We get an $s+1$ by $s+1$ submatrix Δ of $\mathcal{M}$ by using as well the last row of $\mathcal{M}$ and the appropriate entries from the last column of $\mathcal{M}$. If we calculate $\det(\Delta)$ by expanding with respect to the last row, we get, up to sign, $u\det(\Delta_0)$. This shows that $I_s(\mathcal{M}_0) \subseteq I_{s+1}(\mathcal{M})$. $\square$

This completes the argument that the Jacobian ideal $\mathcal{J}_{R/A}$ is independent of the presentation of R over A.

We next want to observe what happens to the Jacobian ideal when we localize S at one (or, equivalently, at finitely many) elements. Consider what happens when we localize at $u \in R$, where u is the image of $h(X_1, \ldots, X_s) \in A[X_1, \ldots, X_s]$, where we have chosen an A-algebra surjection $A[X_1, \ldots, X_s] \twoheadrightarrow R$. We may use $1/u$ as an additional generator, and introduce a new variable X_{s+1} that maps to $1/u$. We only need one additional equation, $X_{s+1}h(X_1, \ldots, X_s) - 1$, as a generator. The original Jacobian matrix is in the upper left corner of the new Jacobian matrix, and the new bottom row consists of all zeroes except for the last entry, which is $h(X_1, \ldots, X_s)$. Since the image of this entry is u and so invertible in $R[u^{-1}]$, the Lemma above shows that the new Jacobian ideal is generated by the original Jacobian ideal. We have proved:

Proposition. *If R is a finitely presented A-algebra and S is a localization of R at one (or finitely many) elements, $\mathcal{J}_{S/A} = \mathcal{J}_{R/A}S$.* $\square$

If R is essentially of finite type over A, and $R_0 \subseteq R$, $R_1 \subseteq R$ are two finitely generated A-subalgebras such that $R = W_0^{-1}R_0$ and $R = W_1^{-1}R_1$, then $\mathcal{J}_{R_0/A}R = \mathcal{J}_{R_1/A}R$. Thus, we may define $\mathcal{J}_{R/A}$ to be $\mathcal{J}_{R_0/A}R$ for any choice of such an R_0.

To see why $\mathcal{J}_{R_i/A}R$ does not depend on the choice of $i = 0, 1$, first note that each generator of R_1 is in $(R_0)_w$ for some $w \in W_1$. Hence, we can choose $w \in W_1$ such that $R_1 \subseteq (R_0)_w$. Replacing R_0 by $(R_0)_w$ does not affect $\mathcal{J}_{R_0/A}R$, by the Proposition above. Therefore, we can assume that $R_1 \subseteq R_0$. Similarly, we can choose y in W_1 such that each generator of R_0 is in $(R_1)_y$. Then $(R_0)_y = (R_1)_y$ and replacing R_i by $(R_i)_y$ does not affect $\mathcal{J}_{R_i/A}R$. $\square$

We now obtain the existence of test elements.

Theorem (existence of test elements via the Lipman-Sathaye theorem). *Let R be a domain module-finite and generically smooth over the regular domain A of characteristic p. Then every element c of $J = \mathcal{J}(R/A)$ is such that $cR^{1/q} \subseteq A^{1/q}[R]$ for all q, and, in particular, $cR^\infty \subseteq A^\infty[R]$. Thus, if $c \in J \cap R^\circ$, it is a completely stable big test element.*

Moreover, if $A \hookrightarrow B$ is a flat injective homomorphism of A into a regular domain B, the image of c is a completely stable big test element for $B \otimes_A R$.

Proof. Since $A^{1/q}[R] \cong A^{1/q} \otimes_A R$, the image of c is in $\mathcal{J}(A^{1/q}[R]/A^{1/q})$, and so the Lipman-Sathaye theorem implies that c multiplies the normalization S' of $S = A^{1/q}[R]$ into $A^{1/q}[R]$. But $R^{1/q} \subseteq S'$ by the Lemma on p. 1. $\square$

Geometrically reduced K-algebras

Let K be a field, and let R be a K-algebra. For the purpose of this discussion, we do not need to impose any finiteness condition on R.

Proposition. *The following conditions on R are equivalent.*

(1) *For every finite purely inseparable extension L of K, $L \otimes_K R$ is reduced.*

(2) *$K^\infty \otimes_K R$ is reduced, where K^∞ is the perfect closure of K.*

(3) *$\overline{K} \otimes_K R$ is reduced, where $\overline{K}$ is an algebraic closure of K.*

(4) *For some field L containing K^∞, $L \otimes_K R$ is reduced.*

(5) *For every field extension L of K, $L \otimes_K R$ is reduced.*

Proof. If $K \subseteq L \subseteq L'$ are field extensions, then $L' \otimes_K R = L' \otimes_L (L \otimes_K R)$ is free and, in particular, faithfully flat over R. Hence, $L \otimes_K R \subseteq L' \otimes_K R$, and $L \otimes_K R$ is reduced if $L' \otimes_K R$ is reduced. Hence (5) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (1), while (1) $\Rightarrow$ (2) because $K^\infty \otimes_K R$ is the directed union of the rings $L \otimes_K R$ as L runs though the subfields of K^∞ that are finite algebraic over R. Thus, it will suffice to show that (2) $\Rightarrow$ (5).

For a fixed field extension L of K, $L \otimes_K R$ is reduced if and only if $L \otimes_K R_0$ is reduced for every finitely generated K-subalgebra R_0 of R. Hence, to prove the equivalence of (2) and (5), it suffices to consider the case where R is finitely generated over K. In either case, R itself is reduced. Let W be the multiplicative system of all nonzerodivisors in R. These are also nonzerodivisors in $L \otimes_K R$, and $W^{-1}(L \otimes_K R) \cong L \otimes_K (W^{-1}R)$. Therefore, in proving the equivalence of (2) and (5), we may replace R by its total quotient ring, which is a product of fields finitely generated as fields over K. Thus, we may assume without loss of generality that R is a field finitely generated over K. In this case, $L \otimes_K R$ is a zero-dimensional Noetherian ring, and so $L \otimes_K R$ is reduced if and only if it is regular, and the result follows from our treatment of geometric regularity: see page 2 of the Lecture Notes from September 19. $\square$

We define R to be *geometrically reduced* over K if it satisfies the equivalent conditions of this Proposition. If R is essentially of finite type over K or if R is Noetherian and L is a finitely generated field extension of K, $L \otimes_K R$ will be Noetherian. Our main interest is in the case where R is finitely generated over K.

The Jacobian ideal of a finitely generated K-algebra

Let R be a finitely generated K-algebra such that the quotient by every minimal prime has dimension d. Suppose that $R = K[x_1, \ldots, x_n]/I$. Then we define the Jacobian ideal $\mathcal{J}_{R/K}$ as the ideal generated by the images of the $n-d$ size minors of the matrix $\big(\partial f_i/\partial x_j\big)$ in R. This ideal is independent of the choice of presentation: the argument is the same as in the treatment of $\mathcal{J}_{R/A}$ on p. 3. If K is algebraically closed, this ideal defines the singular locus. We give the argument.

Theorem. *Let K be an algebraically closed field. Let R be a finitely generated K-algebra such that the quotient by every minimal prime has dimension d. For a given prime ideal P of R, R_P is regular if and only if P does not contain $\mathcal{J}_{R/K}$.*

Proof. We first consider the case where $P = m$ is maximal. By adjusting the constant terms, we can pick generators for R that are in m. Then we can write $R = K[x_1, \ldots, x_n]/I$ and assume without loss of generality that $m = (x_1, \ldots, x_n)R$. Let $f_1, \ldots, f_m$ generate I. Then R_m is regular if and only if R_m/m^2R_m has K-vector space dimension d. Since R/m^2 is already local, $R/m^2 \cong R_m/m^2R_m$, and $m/m^2 \cong mR_m/m^2R_m$. The K-vector space dimension of m/m^2 is $n-r$, where r is the K-vector space dimension of the K-span V of the linear forms that occur in the f_j: in fact, m/m^2 is the quotient of $Kx_1 + \cdots + Kx_n$ by V. The K-vector space V is isomorphic with the column space of the matrix

$$\left(\frac{\partial f_j}{\partial x_i}\right)\Big|_{(0,\ldots,0)},$$

where the partial derivative entries are evaluated at the origin: the j th column corresponds to the vector of coefficients of the linear form occurring in f_j. Thus, the ring R_m is regular if and only if the evaluated matrix has rank $n-d$: the rank cannot be larger than $n-d$, since R_m has dimension d. This will be the case if and only if some $n-d$ size minor does not vanish at the origin, and this is equivalent to the statement that $\mathcal{J}_{R/K}$ is not contained in m.

In the general case, if P does not contain J, let $f \in J - P$. Choose a maximal ideal of R_f that contains PR_f. The quotient R_f/PR_f is K, and so this maximal ideal corresponds to a maximal ideal m of R containing P and not containing f. Since R_m is regular, so is its localization R_P.

Now suppose that P contains $\mathcal{J}_{R/K}$ but R_P is regular. Then PR_P is generated by height (P) elements, and we can choose $f \in R-P$ such that PR_f is generated by height (P) elements. Choose a maximal ideal of $(R/P)_f$ such that the localization at this maximal ideal is regular. This will correspond to a maximal ideal m of R containing P and not f. Since PR_m is generated by height (P) elements (f is inverted in this ring) and $m(R/P)_m$

is generated by by $\dim\big((R/P)_m\big)$ elements, mR_m is generated by $\dim(R_m)$ elements, and so R_m is regular. But this contradicts $\mathcal{J}_{R/K} \subseteq P \subseteq m$. $\square$

We want to use the Lipman-Sathaye Theorem to produce specific test elements for finitely generated algebras over a field. We need some preliminary results.

Lemma. *Let $X \to Y$ be a morphism of affine varities over an algebraically closed field K such that the image of X is dense in Y. Then the image of every Zariski dense open subset U of X contains a dense Zariski open subset of Y.*

Proof. The morphism corresponds to a map of domains finitely generated over K. If the map has a kernel P, the image of the morphism would be contained in $\mathcal{V}(P)$, a proper closed set. Hence, we have an injection of domains $A \to B$, and B is finitely generated over K and, hence, over A. U contains an open subset of X of the form $X - \mathcal{V}(f)$, where $f \neq 0$, and so we might as well assume that $U = X - \mathcal{V}(f)$. This amounts to replacing B by B_f, which is still finitely generated over A. After localizing A at one element $a \in A - \{0\}$, we have that B_a is a module-finite extension of a polynomial ring C over A_a, by Noether normalization over a domain. Then $\mathrm{Spec}\,(B_a) \to \mathrm{Spec}\,(C)$ and $\mathrm{Spec}\,(C) \to \mathrm{Spec}\,(A_a)$ are both surjective, so that the image of the morphism contains $Y - \mathcal{V}(a)$. $\square$

Theorem. *Let K be an algebraically closed field and R a finitely generated K-algebra of dimension d. Let $\underline{x} = x_1, \ldots, x_n$ be generators of R over K, and let $\mathrm{GL}(n, K)$ act so that γ replaces these generators by the entries of the column*

$$\gamma \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

We use $\gamma(\underline{x})$ to denote this sequence of n elements of R, which also generates R over K, and use $\gamma_i(\underline{x})$ to denote its i th entry.

(a) *There is a dense Zariski open subset U of $\mathrm{GL}(n, K)$ such that for all $\gamma \in U$, R is module-finite over the ring generated over K by $\gamma_{i_1}(\underline{x}), \ldots, \gamma_{i_d}(\underline{x})$ for every choice of d mutually distinct indices $i_1, \ldots, i_d$ in $\{1, \ldots, n\}$. Moreover, for every choice of the d elements, $\gamma_{i_1}(\underline{x}), \ldots, \gamma_{i_d}(\underline{x})$ are algebraically independent over K.*

(b) *Assume in, addition, that R is reduced, and that the quotient of R by every minimal prime has dimension d. Then there is a dense Zariski open subset U_0 of $\mathrm{GL}(n, K)$ such that, in addition, R is torsion-free and generically étale over the polynomial ring $K[\gamma_{i_1}(\underline{x}), \ldots, \gamma_{i_d}(\underline{x})]$ for every choice of $i_1, \ldots, i_d$ in $\{1, \ldots, n\}$.*

Proof. (a) This is a variant of Noether normalization. It suffices to show that there is a dense Zariski open subset such that R is module-finite over $K[\gamma_1(\underline{x}), \ldots, \gamma_d(\underline{x})]$. By symmetry, we obtain a dense open subset for each of the $\binom{n}{d}$ possible choices of $i_1, \dots, i_d$, and we may intersect them all.

In this paragraph we allow K to be any infinite field, not necessarily algebraically closed. Note that if $n > d$ there is a nontrivial polynomial relation $F(X_1, \dots, X_n)$ on $x_1, \dots, x_n$ over K, i.e., $F(x_1, \dots, x_n) = 0$. Let $\underline{y} = y_1, \dots, y_n$ be the image of $\underline{x}$ under γ. Then $\delta = \gamma^{-1}$ maps $\underline{y}$ to $\underline{x}$, and the elements $\underline{y}$ have the relation $F\bigl(\delta(\underline{Y})\bigr) = 0$. Let $h = \deg(F)$ and let G be the degree $h > 0$ form in F. The key point is that for δ in a dense open set of $\mathrm{GL}(n, K)$ ($\gamma = \delta^{-1}$ will also vary in a dense open set), $G\bigl(\delta(\underline{Y})\bigr)$ will be monic in every Y_i. To see this for, for example, Y_n, it suffices to see that G does not become 0 when we specialize all the Y_j for $j < n$ to 0. But this yields a power of Y_n times the value of G on the nth column of δ, and the polynomial G obviously vanishes only on a proper closed subset of $\mathrm{GL}(n, K)$. Hence, for every j, y_j is integral over the subring generated by the other y_i.

From now on we assume that K itself is algebraically closed. If we enlarge K by adjoining n^2 algebraically indeterminates $t_{ij}^{(1)}$, and let $K_1 = K(t_{ij}^{(1)} : i, j)$, we may carry through the procedure of the preceding paragraph over K_1, letting the matrix $\bigl(t_{ij}^{(1)}\bigr)^{-1}$ act, so that $\delta = \bigl(t_{ij}^{(1)}\bigr)$. It is clear that G does not vanish on the nth column of δ, which is a matrix of indeterminates. Let $\underline{y}^{(1)}$ denote the image of $\underline{x}$.

If $n-1 > d$, we continue in this way, next adjoining $(n-1)^2$ indeterminates $t_{ij}^{(2)}$ to K_1 to produce a field K_2, and letting the matrix $\bigl(t_{ij}^{(2)}\bigr)^{-1}$ act. This matrix is in $\mathrm{GL}(n-1, K_2)$, but we view it as an element of $\mathrm{GL}(n, K_2)$ by taking its direct sum with a 1×1 identity matrix. Thus, it acts on $\underline{y}^{(1)}$ to produce a new sequence of generators $\underline{y}^{(2)}$ for $K_2 \otimes_K R$ over K_2. We now have that $y_{n-1}^{(2)}$ and $y_n^{(2)}$ are integral over the ring $K_2[y_1^{(2)}, \dots, y_{n-2}^{(2)}]$.

We iterate this procedure to produce a sequence of fields K_h and sets of generators $\underline{y}^{(h)}$, $1 \le h \le n-d$. Once K_h and $\underline{y}^{(h)}$ have been constructed so that $K_h \otimes_K R$ is integral over $K[y_1^h, \dots, y_{n-h}^h]$, we construct the next field and set of generators as follows. Enlarge the field K_h to K_{h+1} by introducing $(n-h+1)^2$ new indeterminates $t_{ij}^{(h+1)}$, view the matrix $\bigl(t_{ij}^{(h+1)}\bigr)^{-1}$, which is *a priori* in $\mathrm{GL}(n-h+1, K_{h+1})$, as an element of $\mathrm{GL}(n, K_{h+1})$ by taking its direct sum with an identity matrix of size $h-1$, and let its inverse act on $\underline{y}^{(h)}$ to produce $\underline{y}^{(h+1)}$. We will then have that $K_{h+1} \otimes_K R$ is integral over $K_{h+1}[y_1^{(h+1)}, \dots, y_{n-h-1}^{(h+1)}]$. Thus, we eventually construct a field K_{n-d} such that $K_{n-d} \otimes_K R$ is module-finite over $K_{n-d}[y_1^{(n-d)}, \dots, y_d^{(n-d)}]$. Let $K[\underline{t}]$ denote the polynomial ring in all the indeterimates $t_{ij}^{(h)}$ that we have adjoined to K in forming K_{n-d}. Thus, K_{n-d} is the fraction field $K(\underline{t})$ of the polynomial ring $K[\underline{t}]$. It follows that each of the original generators x_j satisfies a monic polynomial with coefficients in $K(\underline{t})[y_1^{(n-d)}, \dots, y_d^{(n-d)}]$. We can choose a single polynomial $H = H(\underline{t}) \in K[\underline{t}] - \{0\}$ that is a common denominator for all of the coefficients in $K(\underline{t})$ occurring in these monic equations of integral dependence for the various x_j, and such that all the $y_i^{(h)}$ are elements of $K[\underline{t}][1/H] \otimes_K R$. It follows that $K[\underline{t}][1/H] \otimes_K R$ is module-finite over $K[\underline{t}][1/H][y_1^{(n-d)}, \dots, y_d^{(n-d)}]$.

We want to specialize the variables $t_{ij}^{(h)}$ to elements of K in such a way that, first, the matrices $\bigl(t_{ij}^{(h)}\bigr)$ specialize to invertible matrices, and, second, H does not vanish. Such values of $t_{ij}^{(h)}$ correspond to points over K of the open set in

$$G = \mathrm{GL}(n-d+1,\, K) \times \cdots \times \mathrm{GL}(n,\, K)$$

where H does not vanish. (Here, each $\mathrm{GL}(s,\, K)$ for $s \leq n$ is thought of a subgroup of $\mathrm{GL}(n,\, K)$ by identifying $\eta \in \mathrm{GL}(s,\, K)$ with the direct sum of η and a size $n-s$ identity matrix.) Call this point $(\gamma^{(n-d+1)}, \dots, \gamma^{(n)})$. For any such point, the result of specializing all of the $t_{ij}^{(h)}$ in the equations of integral dependence for the x_j shows that R is module-finite over the ring generated over K by the images of $y_1^{n-d}, \dots, y_d^{n-d}$. These images are the first d coordinates of $\gamma(\underline{x})$, where $\gamma = \gamma^{(n-d+1)} \cdots \gamma^{(n)}$. The map of $G \to \mathrm{GL}(n,\, K)$ is surjective. By the preceding Lemma, the image of $G - \mathcal{V}(H)$ contains a dense open subset U of $\mathrm{GL}(n,\, K)$, which completes the proof of part (a).

(b) Since R has pure dimension d, when it is represented as a finite module over a polynomial ring of dimension d, it must be torsion-free: if there were a torsion element u, Ru would be a submodule of R of dimension strictly smaller than d, and R would have an associated prime Q with $\dim(R/Q) < d$. But R is reduced, so that all associated primes are minimal, and these have quotients of dimension d.

To show that there is an open set U_0 such that R is generically étale over every $K[\gamma_{i_1}(\underline{x}), \dots, \gamma_{i_d}(\underline{x})]$ it suffices to show this for $R/\mathfrak{p}$ for each minimal prime $\mathfrak{p}$ of R: we can then intersect the finitely many open sets for the various minimal primes. Thus, we may assume that R is a domain of dimension d.

We want each d element subset of the image of $x_1, \dots, x_n$, call it $y_1, \dots, y_n$, to be such that R is generically étale over every polynomial ring in the d elements of the chosen subset. It suffices to get such an open set for $y_1, \dots, y_d$: by symmetry, there will be an open set of every d element subset of $y_1, \dots, y_n$, and we may intersect these. This condition is precisely that $y_1, \dots, y_d$ be a separating transcendence basis for the fraction field $\mathcal{L}$ of R over K. The fact that K is algebraically closed implies that $\mathcal{L}$ has some separating transcendence basis over K, by the Theorem on p. 4 of the Lecture Notes from September 19. There are now several ways to argue. To give a specific one, we may apply, for example, Theorem 5.10 (d) of [E. Kunz, *Kähler differentials*, Friedr. Vieweg & Sohn, Braunschweig, 1986], which asserts that a necessary and sufficient condition for $y_1, \dots, y_d$ to be a separating transcendence basis is that the differentials of these elements $dy_1, \dots, dy_d$ in the module of Kähler differentials $\Omega_{\mathcal{L}/K} \cong \mathcal{L}^d$ be a basis for $\Omega_{\mathcal{L}/K}$ as an $\mathcal{L}$-vector space. Since the differentials of the original variables span $\Omega_{\mathcal{L}/K}$ over $\mathcal{L}$, it is clear that the set of elements of $\mathrm{GL}(n,\, K)$ for which all d element subsets of the new variables have differentials that span $\Omega_{\mathcal{L}/K}$ contains a Zariski dense open set. $\square$

Theorem (test elements for affine K-algebras via the Lipman-Sathaye theorem). *Let K be a field of characteristic p and let R be a finitely generated d-dimensional geometrically reduced K-algebra such that the quotient by every minimal prime has dimension d. Then $\mathcal{J}_{R/K}$ is generated by its intersection with R°, and the elements in this intersection are completely stable big test elements for R.*

Proof. Let L be an algebraic closure of K. From the definition, $\mathcal{J}_{R/K}$ expands to give $\mathcal{J}_{(L\otimes_K R)/L}$. Since $L \otimes_K R$ is reduced, its localization at any minimal prime is a field and, in particular, is regular. It follows that $\mathcal{J}_{R/K}$ expands to an ideal of $L \otimes_K R$ that is not contained in any minimal prime of $L \otimes_K R$, and so $\mathcal{J}_{R/K}$ cannot be contained in a minimal prime of R. It follows from the Lemma on p. 10 of the Lecture Notes from September 17 that it is generated by its intersection with R°. To show that the specified elements are completely stable big test elements, it suffices to prove this after making a base change to $L \otimes_K R$, by part (b) of the Proposition at the bottom of p. 8 of the Lecture Notes from September 17. Hence, we may assume without loss of generality that K is algebraically closed.

The calculation of the Jacobian ideal is independent of the choice of indeterminates. We are therefore free to make a linear change of coordinates, which corresponds to choosing an element of $G = GL(n, K) \subseteq K^{n^2}$ to act on the one-forms of $K[x_1, \ldots, x_n]$. For a dense Zariski open set U of $G \subseteq K^{n^2}$, if we make a change of coordinates corresponding to an element $\gamma \in U \subseteq G$ then, for every choice of d of the (new) indeterminates, if A denotes the K-subalgebra of R that these d new indeterminates generate, by parts (a) and (b) of the Theorem above, the two conditions listed below will hold:

(1) R is module-finite over A (and the d chosen indeterminates will then, per force, be algebraically independent) and

(2) R is torsion-free and generically étale over A.

Now suppose that a suitable change of coordinates has been made, and, as above, let A be the ring generated over K by some set of d of the elements x_i. Then the $n-d$ size minors of $(\partial f_j/\partial x_i)$ involving the $n-d$ columns of $(\partial f_j/\partial x_i)$ that correspond to the variables not chosen as generators of A precisely generate $\mathcal{J}_{R/A}$. The result is now immediate from the Theorem on p. 8: as we vary the set of d variables, so that A varies as well, every $n-d$ size minor occurs as a generator of some $\mathcal{J}(R/A)$ $\square$