Math 711: Lecture of November 19, 2007

Summary of Local Cohomology Theory

The following material and more was discussed in seminar but not in class. We give a summary here.

Let I be an ideal of a Noetherian ring R and let M be any R-module, not necessarily finitely generated. We define

$$H_I^j(M) = \varinjlim_t \mathrm{Ext}_R^j(R/I^t,\, M).$$

This is called the i th *local cohomology module of* M *with support in* I.

$$H_I^0(M) = \varinjlim_t \mathrm{Hom}_R(R/I^t,\, M)$$

which may be identified with $\bigcup_t \mathrm{Ann}_M I^t \subseteq M$. Every element of $H_I^j(M)$ is killed by a power of I. Evidently, if M is injective then $H_I^j(M) = 0$ for $j \geq 1$. By a taking a direct limit over t of long exact sequences for Ext, we see that if

$$0 \to M' \to M \to M'' \to 0$$

is exact there is a functorial long exact sequence for local cohomology:

$$0 \to H_I^0(M') \to H_I^0(M) \to H_I^0(M'') \to \cdots \to H_I^j(M') \to H_I^j(M) \to H_I^j(M'') \to \cdots .$$

It follows that $H_I^j(\_\,)$ is the j th right derived functor of $H_I^0(\_\,)$. In the definition we may use instead of the ideals I^t any decreasing sequence of ideals cofinal with the powers of I. It follows that if I and J have the same radical, then $H_I^i(M) \cong H_J^i(M)$ for all i.

Theorem. *Let M be a finitely generated module over the Noetherian ring R, and I and ideal of R. Then $H_I^i(M) \neq 0$ for some i if and only if $IM \neq M$, in which case the least integer i such that $H_I^i(M) \neq 0$ is* $\mathrm{depth}_I M$.

Proof. $IM = M$ iff $I + \mathrm{Ann}_R M = R$, and every element of every $H_I^j(M)$ is killed by some power I^N of I and by $\mathrm{Ann}_R M$: their sum must be the unit ideal, and so all the local cohomology vanishes in this case.

Now suppose that $IM \neq M$, so that the depth d is a well-defined integer in $\mathbb{N}$. We use induction on d. If $d = 0$, some nonzero element of M is killed by I, and so $H^0_I(M) \neq 0$. If $d > 0$ choose an element $x \in I$ that is not a zerodivisor on M, and consider the long exact sequence for local cohomology arising from the short exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0.$$

From the induction hypothesis, $H^j_I(M/xM) = 0$ for $j < d-1$ and $H^{d-1}_I(M/xM) \neq 0$. The long exact sequence therefore yields the injectivity of the map

$$H^{j+1}_I(M) \xrightarrow{x} H^{j+1}_I(M)$$

for $j < d-1$. But every element of $H^{j+1}_I(M)$ is killed by a power of I and, in particular, by a power of x. This implies that $H^{j+1}_I(M) = 0$ for $j < d-1$. Since

$$0 = H^{d-1}_I(M) \to H^{d-1}_I(M/xM) \to H^d_I(M)$$

is exact, $H^{d-1}_I(M/xM)$, which we know from the induction hypothesis is not 0, injects into $H^d_I(M)$. $\square$

If $A^\bullet$ and $B^\bullet$ are two right complexes of R-modules with differentials d and d', the *total tensor product* is the right complex whose n th term is

$$\bigoplus_{i+j=n} A^i \otimes_R B^j$$

and whose differential d'' is such that $d''(a_i \otimes b_j) = d(a_i) \otimes b_j + (-1)^i a_i \otimes d'(b_j)$.

Now let $\underline{f} = f_1, \ldots, f_n$ generate an ideal with the same radical as I. Let $\mathcal{C}^\bullet(\underline{f}^\infty;\, R)$ denote the total tensor product of the complexes $0 \to R \to R_{f_j} \to 0$, which gives a complex of flat R-modules:

$$0 \to R \to \bigoplus_j R_{f_j} \to \bigoplus_{j_1<j_2} R_{f_{j_1}f_{j_2}} \to \cdots \to \bigoplus_{j_1<\cdots<j_t} R_{f_{j_1}\cdots f_{j_t}} \to \cdots \to R_{f_1\cdots f_n} \to 0.$$

The differential restricted to R_g where $g = f_{j_1}\cdots f_{j_t}$ takes u to the direct sum of its images, each with a certain sign, in the rings $R_{gf_{j_{t+1}}}$, where j_{t+1} is distinct from $j_1, \ldots, j_t$.

Let

$$\mathcal{C}^\bullet(\underline{f}^\infty;\, M) = \mathcal{C}^\bullet(\underline{f}^\infty;\, R) \otimes_R M,$$

which looks like this:

$$0 \to M \to \bigoplus_j M_{f_j} \to \bigoplus_{j_1<j_2} M_{f_{j_1}f_{j_2}} \to \cdots \to \bigoplus_{j_1<\cdots<j_t} M_{f_{j_1}\cdots f_{j_t}} \to \cdots \to M_{f_1\cdots f_n} \to 0.$$

We temporarily denote the cohomology of this complex as $\mathcal{H}^\bullet_{\underline{f}}(M)$. It turns out to be the same, functorially, as $H^\bullet_I(M)$. We shall not give a complete argument here but we note several key points. First,

$$\mathcal{H}^0_{\underline{f}}(M) = \mathrm{Ker}\,(M \to \bigoplus_j M_{f_j})$$

is the same as the submodule of M consisting of all elements killed by a power of f_j for every j, and this is easily seen to be the same as $H^0_I(M)$. Second, by tensoring a short exact sequence of modules

$$0 \to M' \to M \to M'' \to 0$$

with the complex $\mathcal{C}^\bullet(\underline{f}^\infty;\, R)$ we get a short exact sequence of complexes. This leads to a functorial long exact sequence for $\mathcal{H}^\bullet_{\underline{f}}(\_)$. These two facts imply an isomorphism of the functors $H^\bullet_I(\_)$ and $\mathcal{H}^\bullet_{\underline{f}}(\_)$ provided that we can show that $\mathcal{H}^j_{\underline{f}}(M) = 0$ for $j \geq 1$ when M is injective. We indicate how the argument goes, but we shall assume some basic facts about the structure of injective modules over Noetherian rings.

First note that if one has a map $R \to S$ and an S-module M, then if $\underline{g}$ is the image of $\underline{f}$ in S, we have $H^\bullet_{\underline{f}}(M) = H^\bullet_{\underline{g}}(M)$. This has an important consequence for local cohomology once we establish that the two theories are the same: see the Corollary below.

Every injective module over a Noetherian ring R is a direct sum of injective hulls $E(R/P)$ for various primes P. $E(R/P)$ is the same as the injective hull of the reisdue class field of the local ring R_P. This, we may assume without loss of generality that $(R,\, m,\, K)$ is local and that M is the injective hull of K. This enables to reduce to the case where M has finite length over R, and then, using the long exact sequence, to the case where $M = K$, since M has a finite filtration such that all the factors are K. Thus, we may assume that $M = K$. The complex $\mathcal{C}^\bullet(\underline{f}^\infty;\, R)$ is then a tensor product of complexs of the the form $0 \to R \to R \to 0$ and $0 \to R \to 0 \to 0$. If we have only the latter the complex has no terms in higher degree, while if there are some of the former we get a cohomogical Koszul complex $\mathcal{K}^\bullet(g_1, \ldots, g_n; K)$ where at least one $g_j \neq 0$. But then $(g_1, \ldots, g_n)K = K$ kills all the Koszul cohomology. Thus, we get vanishing of higher cohomology in either case. It follows that $\mathcal{H}^\bullet_{\underline{f}}(\_)$ and $H^\bullet_I(\_)$ are isomorphic functors, and we drop the first notation, except in the proof of the Corollary just below.

Corollary. *If $R \to S$ is a homomorphism of Noetherian rings, M is an S-module, and ${}_RM$ denotes M viewed as an R-module via restriction of scalars, then for every ideal I of R, $H^\bullet_I({}_RM) \cong H^\bullet_{IS}(M)$.*

Proof. Let $f_1, \ldots, f_n$ generate I, and let $g_1, \ldots, g_n$ be the images of these elements in S: they generate IS. Then we have

$$H^\bullet_I({}_RM) \cong \mathcal{H}^\bullet_{\underline{f}}({}_RM) \cong \mathcal{H}^\bullet_{\underline{g}}(M) \cong H^\bullet_{IS}(M)\,. \qquad \square$$

We note that the complex $0 \to R \to R_f \to 0$ is isomorphic to the direct limit of the cohomological Koszul complexes $\mathcal{K}^\bullet(f^t; R)$, where the maps between consecutive complexes are given by the identity on the degree 0 copy of R and by multiplication by f on the degree 1 copy of R — note the commutativity of the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{f^{t+1}} & R & \longrightarrow & 0 \\ & & \mathrm{id}\uparrow & & \uparrow f & & \\ 0 & \longrightarrow & R & \xrightarrow{f^t} & R & \longrightarrow & 0 \end{array}.$$

Tensoring these Koszul complexes together as f runs through $f_1, \ldots, f_n$, we see that

$$\mathcal{C}^\bullet(\underline{f}^\infty; M) = \varinjlim{}_t \, \mathcal{K}^\bullet(f_1^t, \ldots, f_n^t; M).$$

Hence, whenever $f_1, \ldots, f_n$ generate I up to radicals, taking cohomology yields

$$H_I^\bullet(M) \cong \varinjlim{}_t \, H^\bullet(f_1^t, \ldots, f_n^t; M).$$

When R is a local ring of Krull dimension d and $x_1, \ldots, x_d$ is a system of parameters, this yields

$$H_m^d(R) = \varinjlim{}_t \, R/(x_1^t, \ldots, x_d^t)R.$$

Likiewise, for every R-module M,

$$H_m^d(M) = \varinjlim{}_t \, M/(x_1^t, \ldots, x_d^t)M \cong H_m^d(R) \otimes_R M.$$

We next recall that when (R, m, K) is a complete local ring and $E = E_R(K)$ is an injective hull of the residue class field (this means that $K \subseteq E$, where E is injective, and every nonzero submodule of E meets K), there is duality between modules with ACC over R and modules with DCC: if M satsifies one of the chain conditions then $M^\vee = \mathrm{Hom}_R(M, E)$ satisfies the other, and the canonical map $M \to M^{\vee\vee}$ is an isomorphism in either case. In particular, when R is complete local, the obvious map $R \to \mathrm{Hom}_R(E, E)$ is an isomorphism. An Artin local ring R with a one-dimensional socle is injective as a module over itself, and, in this case, $E_R(K) = R$. If R is Gorenstein and $x_1, \ldots, x_d$ is a system of parameters, one has that each $R_t = R/(x_1^t, \ldots, x_d^t)R$ is Artin with a one-dimensional socle, and one can show that in this case $E_R(K) \cong H_M^d(R)$. When R is local but not complete, if M has ACC then $M^\vee$ has DCC, and $M^{\vee\vee}$ is canonically isomorphic with $\widehat{M}$. If M has DCC, $M^\vee$ is a module with ACC over $\widehat{R}$, and $M^{\vee\vee}$ is canonically isomorphic with M.

We can make use of this duality theory to gain a deeper underrstanding of the behavior of local cohomology over a Gorenstein local ring.

Theorem (local duality over Gorenstein rings). *Let $(R,\, m,\, K)$ be a Gorenstein local ring of Krull dimension d, and let $E = H^d_m(R)$, which is also an injective hull for K. Let M be a finitely generated R-module. Then for every integer j, $H^j_m(M) = \mathrm{Ext}^{d-j}_R(M,\, R)^\vee$.*

Proof. Let $x_1, \, \ldots\, , x_d$ be a system of parameters for R. In the Cohen-Macaulay case, the local cohomology of R vanishes for $i < d$, and so $\mathcal{C}^\bullet(\underline{x}^\infty; R)$, numbered backwards, is a flat resolution of E. Thus,

$$H^j_m(M) \cong \mathrm{Tor}^R_{d-j}(M,\, E).$$

Let $G_\bullet$ be a projective resoultion of M by finitely generated projective R-modules. Then

$$\mathrm{Ext}^{d-j}_R(M,\, R)^\vee \cong H^{d-j}\bigl(\mathrm{Hom}_R(G_\bullet,\, R),\, E\bigr)$$

(since E is injective, $\mathrm{Hom}_R(\_\,,\, E)$ commutes with the calculation of cohomology). The functor $\mathrm{Hom}_R\bigl(\mathrm{Hom}_R(\_\,,\, R),\, E\bigr)$ is isomorphic with the functor $\_ \otimes E$ when restricted to finitely generated projective modules G. To see this, observe that for every G there is an R-bilinear map $G \times E \to \mathrm{Hom}_R\bigl(\mathrm{Hom}_R(G,\, R),\, E\bigr)$ that sends $(g,\, u)$ (where $g \in G$ and $u \in E$) to the map whose value on $f : G \to R$ is $f(g)u$. This map is an isomorphism when $G = R$, and commutes with direct sum, so that it is also an isomorphism when G is finitely generated and free, and, likewise, when G is a direct summand of a finitely generated free module. But then

$$\mathrm{Ext}^{d-j}_R(M,\, R)^\vee \cong H_{d-j}(G_\bullet \otimes E) \cong \mathrm{Tor}^R_{d-j}(M,\, E),$$

which is $\cong H^j_m(M)$, as already observed. $\square$

Corollary. *Let M be a finitely generated module over a local ring $(R,\, m,\, K)$. Then the modules $H^i_m(M)$ have DCC.*

Proof. The issues are unchanged if we complete R and M. Then R is a homomorphic image of a complete regular local ring, which is Gorenstein. The problem therefore reduces to the case where the ring is Gorenstein. By local duality, $H^i_m(M)$ is the dual of the Noetherian module $\mathrm{Ext}^{n-i}_R(M,\, R)$, where $n = \dim(R)$. $\square$

The action of the Frobenius endomorphism on local cohomolgy

Let R be a ring of prime characteristic $p > 0$, and let $I = (f_1,\, \ldots\, , f_n)R$. Consider the complex $\mathcal{C}^\bullet = \mathcal{C}^\bullet(\underline{f}^\infty;\, R)$, which is

$$0 \to R \to \bigoplus_j R_{f_j} \to \bigoplus_{j_1 < j_2} R_{f_{j_1} f_{j_2}} \to \cdots \to \bigoplus_{j_1 < \cdots < j_t} R_{f_{j_1} \cdots f_{j_t}} \to \cdots \to R_{f_1 \cdots f_n} \to 0.$$

This complex is a direct sum of rings of the form R_g each of which has a Frobenius endomorphism $F_{R_g} : R_g \to R_g$. Given any homomorphism $h : S \to T$ of rings of prime characteristic $p > 0$, there is a commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{h} & T \\ {\scriptstyle F_S}\uparrow & & \uparrow{\scriptstyle F_T} \\ S & \xrightarrow{h} & T \end{array}$$

The commutativity of the diagram follows simply because $h(s)^p = h(s^p)$ for all $s \in S$. Since every $\mathcal{C}^i$ is a direct sum of R-algebras, each of which has a Frobenius endomorphism, collectively these endomorphism yield an endomorphism of $\mathcal{C}^i$ that stabilizes every summand and is, at least, $\mathbb{Z}$-linear. This gives an endomorphism of $\mathcal{C}^\bullet$ that commutes with differentials $\delta^i : \mathcal{C}^i \to \mathcal{C}^{i+1}$ in the complex. The point is that the restriction of the differential to a term R_g may be viewed as a map to a product of rings of the form R_{gf}. Each component map is either h or $-h$, where $h : R_g \to R_{gf}$ is the natural localization map, and is a ring homomorphism. The homomorphism h commutes with the actions of the Frobenius endomorphisms, and it follows that $-h$ does as well.

This yields an action of F on the complex and, consequently, on its cohomology, i.e., an action of F on the local cohomology modules $H^i_I(R)$. It is not difficult to verify that this action is independent of the choice of generators for I. This action of F is more than $\mathbb{Z}$-linear. It is easy to check that for all $r \in R$, $F(ru) = r^p F(u)$. This is, in fact, true for the action on $\mathcal{C}^\bullet$ as well as for the action on $H^\bullet_I(R)$.

If $R \to S$ is any ring homomorphism, there is an induced map of complexes

$$\mathcal{C}^\bullet(\underline{f}^\infty;\, R) \to \mathcal{C}^\bullet(\underline{f}^\infty;\, S).$$

It is immediate that the actions of F are compatible with the induced maps of local cohomology, i.e., that the diagrams

$$\begin{array}{ccc} H^i_I(R) & \longrightarrow & H^i_I(S) \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle F} \\ H^i_I(R) & \longrightarrow & H^i_I(S) \end{array}$$

commute.

We now want to use our understanding of local cohomology to prove the Theorem of Huneke and Lyubeznik.

Discussion. We are primarily interested in studying R^+ when $(R,\, m,\, K)$ is a local domain that is a homomorphic image of a Gorenstein ring A. If $\mathcal{M}$ is the inverse image of m in A, we may replace A by $A_{\mathcal{M}}$ and so assume that A is local.

Note, however, that when we take a module-finte extension domain of R, the ring that we obtain is no longer local: it is only semilocal. Therefore, we shall frequently have the hypothesis that R is a semilocal domain that is a module-finite extension of a homomorphic image of a Gorenstein local ring.

Let $(A, \mathfrak{m}, K)$ denote a Gorenstein local ring, $\mathfrak{p}$ a prime ideal of this ring, and R a local domain that is a module-finite extension of $B = A/\mathfrak{p}$. R is semilocal in this situation. The maximal ideals of R are the same as the prime ideals m that lie over $\mathfrak{m}/\mathfrak{p}$, since R/m is a module finite extension of $B/(m \cap B)$, and so R/m has dimension 0 if and only if $B/(m \cap B)$ has dimension 0, which occurs only when $m \cap B$ is the maximal ideal $\mathfrak{m}/\mathfrak{p}$ of B. Note that since A is Gorenstein, it is Cohen-Macaulay, and therefore universally catenary. Hence, so is R. The Jacobson radical $\mathfrak{A}$ of R will be the same as the radical of $\mathfrak{m}R$.

By the dimension formula, which is stated on p. 3 of the Lecture Notes from September 18 for Math 711, Fall 2006, and proved in the Lecture Notes from September 20 from the same course on pp. 3–5, we have that $\text{height}\,(m) = \text{height}\,(\mathfrak{m}/\mathfrak{p})$ for every maximal ideal m of R: thus, the height of every maximal ideal is the same as $\dim(R) = \dim(A/\mathfrak{p})$.

We next observe the following fact:

Proposition. *Let R be a domain and let W be a multiplicative system of R that does not contain* 0.

(a) *If T is an extension domain of $W^{-1}R$ and $u \in T$ is integral over $W^{-1}R$, then there exists $w \in W$ such that wu is integral over R.*

(b) *If T is module-finite (respectviely, integral) extension domain of R then there exists a module-finite (respectively, integral) extension domain S of R within T such that $T = W^{-1}S$.*

(c) *If $W^{-1}(R^+)$ is an absolute integral closure for $W^{-1}R$, i.e., we may write $W^{-1}(R^+) \cong (W^{-1}R)^+$.*

(d) *If I is any ideal of R, $I(W^{-1}R)^+ \cap W^{-1}R = W^{-1}(IR^+ \cap R)$. That is, plus closure commutes with localization.*

Proof. (a) Consider an equation of integral dependence for u on T. We may multiply by a common denominator $w \in W$ for the coefficients that occur to obtain an equation

$$wu^k + r_1u^{k-1} + \cdots + r_iu^{k-i} + \cdots + r_{k-1}u + r_k = 0,$$

where the $r_i \in R$. Multiply by w^{k-1}. The resulting equation can be rewritten as

$$(wu)^k + r_1(wu)^{k-1} + \cdots + w^{i-1}r_i(wu)^{k-i} + \cdots + w^{k-2}r_{k-1}(wu) + w^{k-1}r_k = 0,$$

which shows that wu is integral over R, as required.

(b) If T is module-finite over R, choose a finite set of generators for T over R. In the case where T is integral, choose an arbitrary set of generators for T over R. For each generator t_i, choose $w_i \in W$ such that $w_i t_i$ is integral over R. Let S be the extension of R generated by all the $w_i t_i$.

(c) We have that $W^{-1}R^+$ is integral over $W^{-1}R$ and so can be enlarged to a plus closure T. But each element $u \in T$ is integral over $W^{-1}R$, and so has there exists $w \in W$ such thtat wu is integral over R, which means that $wu \in R^+$. But then $u = w^{-1}(wu) \in W^{-1}R^+$, and it follows that $T = W^{-1}R^+$.

(d) Note that $\supseteq$ is obvious. Now suppose that $u \in I(W^{-1}R)^+ = IW^{-1}(R^+)$ by part (c). Then we can choose $w \in W$ such that $wu \in IR^+$, and the result follows. $\square$

Remark. Part (d) may be paraphrased as asserting that plus closure for ideals commutes with arbitrary localization. I.e., $(IW^{-1}R)^+ = W^{-1}(I^+)$. Here, whenever J is an ideal of a domain S, $J^+ = (JS^+) \cap S$.

Remark: plus closure for modules. If R is a domain we can define the plus closure of $N \subseteq M$ as the set of elements of M that are in $\langle R^+ \otimes_R N \rangle$ in $R^+ \otimes_R M$. It is easy to check that the analogue of (d) holds for modules as well.

We are now ready to begin the proof of the following result.

Theorem (Huneke-Lyubeznik). *Let R be a semilocal domain of prime characteristic $p > 0$ that is a module-finite extension of a homomorphic image of a Gorenstein local ring $(A, \mathfrak{m}, K)$. Let $\mathfrak{A}$ denote the Jacobson radical in R, which is the same as the radical of $\mathfrak{m}R$. Let d be the Krull dimension of R. Then there is a module-finite extension domain S of R such that for all $i < d$, the map $H^i_{\mathfrak{m}R}(R) \to H^i_{\mathfrak{m}S}(S)$ is 0. If $\mathfrak{B}$ denotes the Jacobson radical of S, we may rephrase this by saying that $H^i_{\mathfrak{A}}(R) \to H^i_{\mathfrak{B}}(S)$ is 0 for all $i < d$.*

Proof. Let n denote the Krull dimension of the local Gorenstein ring $(A, \mathfrak{m}, K)$. Since R is a module-finite extension of $A/\mathfrak{p}$, we have that the height of $\mathfrak{p}$ is $n - d$.

Recall from the discussion above that $\mathfrak{A}$ (respectively, $\mathfrak{B}$) is the radical of $\mathfrak{m}R$ (respectively, $\mathfrak{m}S$). This justifies the rephrasing. We may think of the local cohomology modules as $H^i_{\mathfrak{m}}(R)$ and $H^i_{\mathfrak{m}}(S)$.

It suffices to solve the problem for one value of i. The new ring S satisfies the same hypotheses as R. We may therefore repeat the process d times, if needed, to obtain a module-finite extension such that all local cohomology maps to 0: once it maps to 0 for a given S, it also maps to 0 for any further module-finite extension. In the remainder of the proof, i is fixed.

It follows from local duality over A that is suffices to choose a module-finite extension S of R such that the map

$$(*) \quad \mathrm{Ext}^{n-i}_A(S, A) \to \mathrm{Ext}^{n-i}_A(R, A)$$

is 0, since the map of local cohomology is the dual of this map. Note that both of the modules in $(*)$ are finitely generated as A-modules. We shall use induction on $\dim(R)$ to reduce to the case where the image of the map has finite length over A: we then prove a theorem to handle that case. Let V_S denote the image of the map.

Let $P_1, \ldots, P_h$ denote the associated primes over A of the image of this map that are not the maximal ideal of A. Note that as S is taken successively larger, the image V_S cannot increase. Also note that since since V_S is a submodule of $N = \mathrm{Ext}^{n-i}_A(R, A)$, any associated prime of V_S is an associated prime of N. We show that for each P_i, we can choose a module-finite extension S_i of R such that P is not an associated prime of V_{S_i}. This will remain true when we enlarge S_i further. By taking S so large that it contains all the S_i, we obtain V_S which, if it is not 0, can only have the associated prime $\mathfrak{m}$. This implies that V_S has finite length over A, as required.

We write P for P_i. Let $W = A - P$. Then $W^{-1}R = R_P$ is module-finite over the Gorenstein local ring A_P. Let P have height s in A, where $s < n$. By local duality over A_P, we have that the dual of $\mathrm{Ext}^{n-i}_{A_P}(M_P, A_P)$ is, functorially, $H^{s-(n-i)}_{PA_P}(M_P)$ for every finitely generated A-module M. Since $i < d$,

$$s - (n - i) < s - (n - d) = s - \mathrm{height}\,(\mathfrak{p}) = s - \mathrm{height}\,(\mathfrak{p}A_P) = \dim\,(A_P/\mathfrak{p}A_P).$$

By the induction hypothesis we can choose a module-finite extension T of R_P such that

$$H^{s-(n-i)}_{PA_P}(T) \to H^{s-(n-i)}_{PA_P}(R_P)$$

is 0. By part (b) of the Proposition on p. 7, we can choose a module-finite extension S of R such that $T = S_P$. Then we have the dual statement that

$$\mathrm{Ext}^{n-i}_{A_P}(S_P, A_P) \to N_P$$

is 0, which shows that $(V_S)_P = 0$. But then P is not an associated prime of V_S, as required.

Thus, we can choose a module-finite extension S of R such that V_S has finite length as an A-module. Taking duals, we find that the image of $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(S)$ has finite length as an A-module, Since Frobenius acts on both of these local cohomology modules so that the action is compatible with this map, it follows that the image W of the map is stable under the action of Frobenius. Moreover, W is a finitely generated A-module, and, consequently, a finitely generated S-module. It suffices to show that we can take a further module-finite extension T of S so as to kill the image of W in $H^i_{\mathfrak{m}}(T)$. This follows from the Theorem below. $\square$

Theorem. *Let $I \subseteq S$ be an ideal of a Noetherian domain S of prime characteristic $p > 0$, and let W be a finitely generated submodule of $H^i_I(S)$ that is stable under the action of the Frobenius endomorphism F. Then there is a module-finite extension T of S such that the image of W in $H^i_I(T)$ is 0.*

Notice that there is no restriction on i in this Theorem. We shall, in fact, prove a somewhat stronger fact of this type.