Math 711: Lecture of November 21, 2007

We are aiming to prove the Theorem stated at the bottom of the last page of the Lecture Notes from November 19, which will complete the proof of the Huneke-Lyubeznik Theorem. We first want to make an observation about local cohomology when the ring is not Noetherian, and then we prove a Lemma that does most of the work.

Discussion. Let R be a ring that is not necessarily Noetherian and let I be an ideal of R that is the radical of a finitely generated ideal. Let M be any R-module. We shall still use the notation $H^i_I(M)$ for

$$H^i\big(\mathcal{C}^\bullet(\underline{f}^\infty;\, M)\big),$$

where $\underline{f} = f_1, \, \ldots \, , f_n$ are elements of R that generate an ideal whose radical is the same as the radical of I. Tht is, we are relaxing the restriction that the base ring be Noetherian.

Note that if R_0 is any subring of R that is finitely generated over the prime ring and contains $f_1, \, \ldots \, , f_n$, then we may view M as an R_0-module, and

$$H^i\big(\mathcal{C}^\bullet(\underline{f}^\infty;\, M)\big) \cong H^i_{(\underline{f})R_0}(M).$$

If $\underline{g} = g_1, \, \ldots \, , g_s$ is another set of elements generating an ideal whose radical is the same as $Rad(I)$, then every f_i has a power in $(g_1, \, \ldots \, , g_s)R$, say

$$f_i^{h_i} = \sum_{j=1}^{s} r_{ij} g_j$$

and every g_j has a power in $(f_1, \, \ldots \, , f_n)R$, say

$$g_j^{k_j} = \sum_{i=1}^{n} r'_{ji} f_i.$$

These equations will evidnetly hold in any subring R_0 of R finitely generated over the prime ring that is sufficiently large to contain $\underline{f}$, $\underline{g}$ and all of the r_{ij} and r'_{ji}. With such a choice of R_0, we see that $H^i_{(\underline{f})R_0}(M) = H^i_{(\underline{g})R_0}(M)$ for all i. Thus, even when R is not Noetherian, this cohomology is independent of the choice of $\underline{f}$.

However, when R is not Noetherian, we do not have available the result that this is the same cohomology theory one gets using Ext.

Notation: polynommial operators in the Frobenius endomorphism. Let R be a ring of prime characteristic $p > 0$, and let $\mathcal{G} = \mathcal{G}(Z)$ be a monic polynomial in one indeterminate Z with coefficients in R, say

$$\mathcal{G} = Z^e + r_1 Z^{e-1} + \cdots + r_{e-1} Z + r_e.$$

Then we may view

$$\mathcal{G}_F = F^e + \cdots + r_1 F^{e-1} + \cdots + r_{e-1}F + r_e\mathbf{1}$$

as an operator on every R-algebra S whose value on $s \in S$ is

$$s^{p^e} + rs^{p^{e-1}} + \cdots + r_1 s^p + r_0 s.$$

$\mathcal{G}_F$ acts on the complexes $\mathcal{C}^\bullet(\underline{f}^\infty; S)$ just as F does: it stabilizes every component summand S_g, and is value on u is

$$F^e(u) + r_1 F^{e-1}(u) + \cdots + r_{e-1}F(u) + r_0 u.$$

Both F and multiplication by an element r of R act on $\mathcal{C}^\bullet(\underline{f}^\infty;\, S)$ so that:

(1) The action is $\mathbb{Z}$-linear.

(2) The action stabilizes every component summand S_g, where g is a product $f_{i_1} \cdots f_{i_h}$.

(3) The action commutes with the differential.

The operators on the complex with these three properties are closed under addition and composition, from which it follows that $\mathcal{G}_F$ is a $\mathbb{Z}$-linear endomorphism of $\mathcal{C}^\bullet(\underline{f}^\infty;\, S)$ that stabilizes every component summand and commutes with the differential. Hence, this operator also acts on the cohomology of the complex.

Lemma. *Let S be a domain of prime characteristic $p > 0$ and $f_1, \ldots, f_n \in S$. Let $I = (f_1, \ldots, f_h)S$. Let $\mathcal{G} = \mathcal{G}(Z)$ be a monic polynomial in one indeterminate Z with coefficients in S. Let $u \in H^i_I(S)$ be such that $\mathcal{G}_F$ kills u. Then S has a module-finite extension domain T such that the image of u under the map $H^i_I(S) \to H^i_I(T)$ is 0.*

Proof. Let $v \in \mathcal{C}^i(\underline{f}^\infty;\, S)$ be a cycle that represents u. Then $\mathcal{G}_F(v)$ is a coboundary, say $\mathcal{G}_F(v) = \delta(w)$, where δ is the diffferential in the complex. Let w_0 be one component of w: it is an element of a ring of the form S_g. The equation

$$\mathcal{G}_F(Y) - w_0 = 0$$

is monic in Y, and so has a solution in a module-finite extension domain of S_g. By part (c) of the Lemma on p. 7 of the Lecture Notes from November 19, there is a module-finite extension domain T_0 of S such that this equation has a solution in $(T_0)_g$. We can find such a module-finite extension for every component summand of $\mathcal{C}^\bullet(\underline{f}^\infty;\, S)$. Since there are only finitely many, we can find a module-finite extension T_1 of S sufficiently large that there is an element w' of $\mathcal{C}^{i-1}(\underline{f}^\infty;\, T_1)$ such that $\mathcal{G}_F(w') = w$. We then have that

$$\mathcal{G}_F(v) = \delta(w) = \delta\big(\mathcal{G}_F(w')\big) = \mathcal{G}_F\big(\delta(w')\big)$$

and so

$$\mathcal{G}_F\big(v - \delta(w')\big) = 0.$$

It follows that every component of $v - \delta(w')$ is a fraction in some $(T_1)_g$ that satisfies a monic polynomial over T_1. Therefore, we may choose a module-finite extension T of T_1 within its fraction field such that all components of $v' = v - \delta(w')$ are in T. It will now suffice to show that v' is a coboundary in $\mathcal{C}^i(\underline{f}^\infty; T)$.

Each component T_g of the complex $\mathcal{C}^\bullet(\underline{f}^\infty; T)$ contains a copy of T. These copies of T form a subcomplex, and this subcomplex contains v'. It will therefore suffice to show that this subcomplex is exact. But this subcomplex is the complex

$$\mathcal{C}^\bullet(\underline{1}^\infty; T)$$

where $\underline{1}$ denotes a string $1,\,1,\,\ldots\,,\,1$ of n elements all of which are 1. Hence, its cohomology is $H^\bullet_T(T)$, which is killed by T and, consequently, is 0. $\square$

We now restate the Theorem we are trying to prove, in a slightly generalized form, and give the argument. In the earlier version, R and S were the same.

Theorem. *Let $I \subseteq S$ be a finitely generated ideal of a domain S of prime characteristic $p > 0$, let $R \subseteq S$ be a Noetherian ring, and and let M be a finitely generated R-submodule of $H^i_I(S)$ that is stable under the action of the Frobenius endomorphism F on $H^i_I(S)$. Then there is a module-finite extension T of S such that the image of M in $H^i_I(T)$ is 0.*

Proof. Let $u \in M$. Consider the ascending chain of R-submodules of M spanned by the initial segments of the sequence

$$u,\, F(u),\, F^2(u),\, \cdots,\, F^k(u),\ ldots.$$

Since M is Noetherian, these submodules stabilize, and so some $F^e(u)$ is an R-linear combination of its predecessors. This yields an equation

$$F^e(u) + s_1 F^{e-1}u + \cdots + s_e u = 0,$$

where the $s_i \in R$. However, the argument makes no further use of this fact.

We may apply the preceding Lemma with

$$G = Z^e + s_1 Z^{e-1} + \cdots + s_e.$$

This shows that there is a module-finite extension T_0 of S such that u maps to 0 in $H^i_I(T_0)$. We can choose such a module-finite extension for every u_j in a finite set of generators $u_1,\,\ldots\,,u_h$ for M over R. We may then choose a module-finite extension T that contains all of these. Then M maps to 0 in $H^i_I(T)$. $\square$

Corollary. *Let R be a semilocal domain of Krull dimension d with Jacobson radical $\mathfrak{A}$ that is module-finite over a Gorenstein local ring. Then $H^i_{\mathfrak{A}}(R^+) = 0$, $0 \le i \le d-1$.*

Proof. $H^i_{\mathfrak{A}}(R^+)$ is the direct limit of the modules $H^i_{\mathfrak{A}}(S)$ as S runs through all module-finite extensions of R. But each $H^i_{\mathfrak{A}}(S)$ maps to 0 in $H^i_{\mathfrak{A}}(T)$ for some further module-finite extension domain T of S, by the Huneke-Lyubeznik Theorem, and so each $H^i_{\mathfrak{A}}(S)$ maps to 0 in $H^i_{\mathfrak{A}}(R^+)$. $\square$

Corollary. *Let $(R,\, m,\, K)$ be a local domain of Krull dimension d that is module-finite over a Gorenstein local ring. Then $H^i_m(R^+) = 0$, $0 \le i \le d-1$.* $\square$

Theorem. *Let $(R,\, m,\, K)$ be a local domain of Krull dimension d that is module-finite over a Gorenstein local ring. Then R^+ is a big Cohen-Macaulay algebra. That is, every system of parameters for R is a regular sequence in R^+.*

Proof. It is clear that $mR^+ \neq R^+$: since R^+ is integral over R, it has a prime ideal that lies over m.

Let $x_1,\, \ldots\,, x_d$ be part of a system of parameters: we must show that it is a regular sequence on R^+. We use induction on $\dim(R)$, and also on d. The result is trivial if $d = 1$, since R^+ is a domain. Assume that $d > 1$ and that we have a counterexample. Then $x_1,\, \ldots\,, x_{d-1}$ is a regular sequence but we can choose $u \in R^+$ such that $ux_d \in (x_1,\, \ldots\,, x_{d-1})R^+$ while $u \notin (x_1,\, \ldots\,, x_{d-1})R^+ = J$. Choose a minimal prime P of R in the support of $(J + Ru)/J$. Then we still have a counterexample when we pass to R_P and $(R_P)^+ \cong (R^+)_P$. By the induction hypothesis we may assume that $P = m$. Then $H^0_m(R^+/J) = 0$.

We can get a contradiction by proving that for every integer h with $0 \le h \le d-1$,

$$H^i_m(R^+/(x_1,\, \ldots\,, x_h)R^+) = 0$$

when $x_1,\, \ldots\,, x_h$ is part of a system of parameters and $i < d - h$. We then have a contradiction, taking $h = d - 1$ in the paragraph just above.

We use induction on h. We already know this when $h = 0$. Now suppose that $S = R^+/(x_1,\, \ldots\,, x_h)R^+$ and that we know that $H^i_m(S) = 0$ for $i < d - h$. Let $x = x_{h+1}$. We want to show that $H^i(S/xS) = 0$ for $i < d - h - 1$. From the short exact sequence

$$0 \to S \xrightarrow{x\cdot} S \to S/xS \to 0,$$

we obtain a long exact sequence part of which is

$$H^i_m(S) \to H^i_m(S/xS) \to H^{i+1}_m(S).$$

For $i < d-h-1$ we also have $i+1 < d-h$, and so both the leftmost term and the rightmost term vanish, which implies that the middle term vanishes as well, as required. $\square$

We shall next expend a considerable effort proving the following Theorem of K. E. Smith:

Theorem. *Let R be a locally excellent Noetherian domain of prime characteristic $p > 0$, and let $x_1,\, \ldots\,, x_d \in R$ be such that $I = (x_1,\, \ldots\,, x_d)R$ has height d. Then $I^* = I^+$.*

We give, in outline, the steps of the proof.

(1) Work with a counterexample with d minimum.

(2) Reduce to the case where R is local, normal, and $x_1, \ldots, x_d$ is a system of parameters. This requires the theorem that R^+ is a big Cohen-Macaulay algebra.

(3) Show that if R is normal local excellent domain, $u \in R$, $I \subseteq R$, and $u \in IT$ for a module-finite extension domain T of $\widehat{R}$, then $u \in IS$ for a module-finite extension domain S of R. This permits a reduction to the case where R is complete. This requires a generalization of Artin approximation that may be deduced from a very difficult Theorem of Popescu.

(4) Reduce to the case where R is Gorenstein.

(5) Let $I_t = (x_1^t, \ldots, x_d^t)R$. Consider $\varinjlim_t I_t^*/I_t$, which may be thought of as 0^* in $H_m^d R)$, as well as $\varinjlim_t I_t^+/I_t$, which may be thought of as 0^+ in $H_m^d(R)$. Also consider the respective annihilators J_* and J_+ of these submodules of $H_m^d(R)$. Show that it suffices to prove that $0^*/0^+$ is 0, and that this module is the Matlis dual of J_+/J_*.

(6) Show that J_+/J_* has finite length. This involves proving that J_* is the test ideal for R, and that formation of the test ideal in an excellent Gorenstein local ring commutes with localization. Then use the induction hypothesis: in proper localizations of R at primes, one knows that tight closure of parameter ideals is the same as plus closure.

(7) Once it is known that J_+/J_* has finite length, it follows that $0^*/0^+$ has finite length, and this module may be identified with the image M of 0^* in $H_m^d(R^+)$. It then follows from the Theorem on p. 3 that M is 0, from which the desired result follows.

Needless to say, it will take quite some time to fill in the details.