Math 711: Lecture of November 28, 2007

Step 3. Reduction to the complete local case. Now suppose that the result holds for ideals of height k of the form $(x_1, \ldots, x_k)R$ whenever $k < d$. Also suppose that (R, m, K) is a normal excellent local ring of prime characteristic $p > 0$ of dimension d, that $I = (x_1, \ldots, x_d)R$ where $x_1, \ldots, x_d$ is a system of parameters for R, and that $u \in I^* - I^+$. We next want to show that there is a counterexample such that R is also complete. Over $\widehat{R}$, we still have that $u \in (I\widehat{R})^*$. If

$$u \in (I\widehat{R})^+,$$

then there is a module-finite extension domain T of $\widehat{R}$ such that $u \in IT$. By the Theorem at the bottom of p. 3 of the Lecture Notes from November 26, there is also a module-finite extension S of R such that $u \in IS$, a contradiction. Henceforth, we may assume that our minimal counterexample is such that R is complete. $\square$

For the next reduction, we need the following fact.

Lemma. *Let A be a normal domain, and let S be a domain extension of A generated by one element u that is integral over A, so that $S = A[u]$. Let $f = f(X)$ be the minimal polynomial of u over $\mathcal{K} = \mathrm{frac}\,(A)$. Then f has coefficients in A, and $S \cong A[x]/(f)$.*

Proof. Let $\deg(f) = n$. Let $\mathcal{L}$ be a splitting field for f over $\mathcal{K}$. Let g be a monic polynomial over A such that $g(u) = 0$. Then $f|g$ working in $\mathcal{K}[X]$. It follows that every root ρ of f satisfies $g(\rho) = 0$. Hence, all of the roots of f in $\mathcal{L}$ are integral over A. We can write

$$f = \prod_{i=1}^{n}(X - \rho_i)$$

where the ρ_i are the roots of f. The coefficients of f are elementary symmetric functions of $\rho_1, \ldots, \rho_n$, and so are integral over A. Since they are also in $\mathcal{K}$ and A is normal, the coefficients of f are in A, i.e., $f \in A[x]$.

Now suppose that $h \in A[X]$ is any polynomial such that $h(u) = 0$. Then $h|f$ working over $\mathcal{K}[X]$. Because f is monic, we can carry out the division algorithm, dividing h by f and obtaining a remainder of degree strictly less than n, entirely over $A[X]$, and the result will be the same as if we had carried out the dvision algorithm over $\mathcal{K}[X]$. Since the remainder is 0 when we carry out the division over $\mathcal{K}[X]$, the remainder is also 0 when we carry out the division over $A[X]$. Consequently, $h \in fA[X]$. It follows that the kernel of the A-algebra surjection $A[X] \twoheadrightarrow A[u] = S$ such that $X \mapsto u$ is precisely $fA[X]$, and the stated result follows. $\square$

Step 4. Reducction the case where R is complete and Gorenstein. Choose a coeffiicient field K for the complete counterexample R. Then R is module-finite over its subring

$A = K[[x_1, \dots, x_d]]$, which is regular, and, in paritcular, normal. Let $S = A[u]$. A domain module-finite over a complete local domain is again local. In S, we still have that

$$u \in \big((x_1, \dots, x_n)S\big)^*,$$

since this becomes true when we make the module-finite extension to R: cf. Problem 4 of Problem Set #4. Moreover, since R is module-finite over S, we may identify $R^+ = S^+$, so that we still have

$$u \notin (x_1, \dots, x_d)S^+.$$

Thus, S also gives a counterexample. By the preceding Lemma, $S \cong A[X]/f$ where f is monic polynomial. Since u is not a unit, the constant term of f is in the maximal ideal of A. It follows that $S \cong A[[X]]/(f)$ as well. Since $A[[X]]$ is regular, $A[[X]]/(f)$ is Gorenstein. We therefore have a minimal counterexample to the Theorem in which the ring is a complete local Gorenstein domain. $\square$

Remark. Until this point in the proof, we have been concerned with keeping R normal. In doing the reduction just above, normality is typically lost. But the remainder of the proof will be carried through for the Gorenstein case, without any further refernce to or need of normality.

We shall soon carry through an investigation that requires the study of the tight closure of 0 in the injective hull of the residue class field of a Gorenstein local ring (R, m, K), which may also be thought of as the highest nonvanishing local cohomology module of the ring with support in m. We shall therefore digress briefly to study some aspects of the behavior $0^*_{H^d_m(R)}$.

Comparison of finitistic tight closure and tight closure

Let R be a Noetherian ring of prime characteristic $p > 0$. When $N \subseteq M$ are modules that are not necessarily finitely generated, we have a notion of tight closure N^*_M.

There is an alternative notion, $N^{*\mathrm{fg}}{}_M$, defined as follows:

$$N^{*\mathrm{fg}}{}_M = \bigcup_{N \subseteq M_0 \subseteq M \text{ with } M_0/N \text{ finitely generated}} N^*_{M_0}.$$

As with tight closure, studying this notion can be reduced to the case where $N = 0$, and in this case

$$0^{*\mathrm{fg}}{}_M = \bigcup_{M_0 \subseteq M \text{ with } M_0 \text{ finitely generated}} 0^*_{M_0}$$

It is not known whether, under mild conditions on R, these two notions are always the same. There has been particularly great interest in the case where the module M is

Artinian, for reasons that we shall discuss in the sequel. The result that ${0^{*\mathrm{fg}}}_M = 0^*_M$ is known in the following cases:

(1) $M = H^d_m(R)$, where $(R,\, m,\, K)$ is a reduced, equidimensional excellent local ring and $d = \dim(R)$.

(2) R is $\mathbb{N}$-graded with $R_0 = K$ and M is a graded Artinian module.

(3) R is excellent, equidimensional reduced local with an isolated singularity, and M is an arbitrary Artinian module.

(4) $(R,\, m,\, K)$ is excellent, local, R_P is Gorenstein if $P \neq m$, and M is the injective hull of the residue class field.

(5) R is excellent local, W is a finitely generated R-module such that W_P has finite injective dimension if $P \neq m$, and M is the Matlis dual of W.

(1) was proved by K. E. Smith, and (2), (3), and (4) by G. Lyubeznik and K. E. Smith. See [G. Lyubeznik and K. E. Smith, *Strong and weak F-regularity are equivalent for graded rings*, Amer. J. Math. **121** (1999), 1279–1290] and [G. Lyubeznik and K. E. Smith, *On the commutation of the test ideal with localization and completion* Trans. Amer. Math. Soc. **353** (2001) 3149–3180]. (5) is a recent result of J. Stubbs in his thesis (University of Michigan, expected May, 2008), whose full results greatly extend (2), (3), and (4), as well as related results in [H. Eltizur, *Tight closure in Artinian modules*, Thesis, University of Michigan, 2003].

We shall prove (1), shortly. We first want to note that the following questions are all open:

Let $(R,\, m,\, K)$ be an excellent local reduced ring of prime characteristic $p > 0$.

(a) Does ${0^{*\mathrm{fg}}}_M = 0^*_M$ in every Artinian R-module M?

(b) Does ${0^{*\mathrm{fg}}}_E = 0^*_E$ in the injective hull E of the residue class field of R?

(c) If R is weakly F-regular, does ${0^{*\mathrm{fg}}}_E = 0^*_E$ in the injective hull of the residue class field? Equivalently, if R is weakly F-regular, is $0^*_E = 0$ in the injective hull E of the residue class field?

(d) If R is weakly F-regular, is R strongly F-regular?

Obviously, an affirmative answer to each of (a), (b), or (c) implies an affirmative answer to the next on the list. Note that in (c), the two formulations are equivalent because in a weakly F-regular ring, $0^{*\mathrm{fg}} = 0$ in every module M, since 0 is tightly closed in every finitely generated module.

What is more, (c) and (d) are equivalent, by the Proposition on p. 3 of the Lecture Notes from October 22.

Hence, an affirmative answer to any one of the statements (a), (b), (c) and (d) implies an affirmative answer to all of the questions following it on the list, while affirmative answers for (c) and (d) are equivalent.