Math 711: Lecture of November 30, 2007

Theorem (K. E. Smith). *Let (R, m, K) be an excellent, reduced, equidimensional local ring of Krull dimension d, and let $H = H^d_m(R)$. Then $0^*_H = 0^{*\mathrm{fg}}_H$. If $x_1, \ldots, x_d$ is a system of parameters for R, $I_t = (x_1^t, \ldots, x_d^t)R$, and $y = x_1 \cdots x_d$, then $y(I_t^*) \subseteq I^*_{t+1}$, and both 0^*_H and $0^{*\mathrm{fg}}_H$ may be thought of as*

$$\varinjlim_t \frac{I_t^*}{I_t},$$

where the map between consecutive terms is induced by multiplication by y on the numerators.

Hence, if (R, m, K) is an excellent, reduced Gorenstein local ring and E is the injective hull of its residue class field, $0^*_E = 0^{*\mathrm{fg}}_E$.

Proof. Note that $yI_t \subseteq I_{t+1}$, and so $(yI_t)^* \subseteq I^*_{t+1}$. In general, for any ideal J, $y(J^*) \subseteq (yJ)^*$, since if $cj^q \in J^{[q]}$, then $c(yj)^q \subseteq y^qJ^{[q]} = (yJ)^{[q]}$. Thus, $y(I_t)^* \subseteq I^*_{t+1}$ for all t.

To prove that $0^*_H = 0^{*\mathrm{fg}}_H$, we need only show $\subseteq$. Let $v \in 0^*_H$. The for some t, v is represented by the class of an element $u \in R$ in R/I_t. We then have that for some $c \in R^\circ$ and for all $q \gg 0$, cv^q is 0 in

$$\mathcal{F}^e(H) \cong \varinjlim_t \frac{R}{(I_t)^{[q]}} \cong H^d_m(R)$$

and this means that for all $q \gg 0$, the element of H represented by the class of cu^q in R/I_{tq} maps to 0 in H. In the generality in which we are working, the maps $R/I_{tq} \to H$ are not necessarily injective. However, this means that for all $q \gg 0$, there exists k_q such that

$$y^{k_q}cu^q \in I_{tq+k_q},$$

and so for all $q \gg 0$,

$$cu^q \in I_{tq+k_q} : y^{k_q}.$$

By the Theorem near the bottom of p. 2 of the Lecture Notes from November 12, the colon ideal on the right is contained in the tight closure of I_{tq}. Note that if $x_1, \ldots, x_d$ were a regular sequence, this colon ideal would be equal to I_{tq}. Hence, for all $q \gg 0$,

$$cu^q \in I^*_{tq}.$$

We may multiply by a test element $c' \in R^\circ$ to obtain that for all $q \gg 0$,

$$c'cu^q \in I_{tq} = (I_t)^{[q]},$$

and so $u \in I_t^*$. This means that the image of u is in the tight closure of 0 in R/I_t, and hence v is in the tight closure of 0 in the image M of R/I_t in H. Hence, $v \in 0_H^{*\mathrm{fg}}$, as required. We have also established the assertion in the final statement of the first paragraph of the Theorem. The assertion in the second paragraph is immediate. $\square$

Remark. Of course, we have a commutative diagram:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \dfrac{R}{I_t} & \xrightarrow{y\cdot} & \dfrac{R}{I_{t+1}} & \longrightarrow \cdots & H_m^d(R) \\
& \uparrow & & \uparrow & & \uparrow \\
\cdots \longrightarrow & \dfrac{I_t^*}{I_t} & \xrightarrow{y\cdot} & \dfrac{I_{t+1}^*}{I_t} & \longrightarrow \cdots & 0^*_{H_m^d(R)}
\end{array}$$

where the vertical arrows are inclusions. Note that if R is Cohen-Macaulay, then all of the arrows, both horizontal and vertical, are inclusions, and so we may think of $0^*_{H_m^d(R)}$ as the ascending union of the modules $\dfrac{I_t^*}{I_t}$. In particular, these remarks apply when R is Gorenstein.

Discussion: the plus closure of 0 in local cohomology. . Let (R, m, K) be a local domain of prime characteristic $p > 0$, and let $d = \dim(R)$. Let $H = H_m^d(R)$. The 0_H^+ is, by definition, the kernel of the map

$$H_m^d(R) \to R^+ \otimes_R H_m^d(R)$$

and the latter is $H_m^d(R^+)$. Let $x_1, \ldots, x_d$ be a system of parameters for R, let $I_t = (x_1^t, \ldots, x_d^t)R$, and let $y = x_1 \cdots x_d$. Then $y(I_t^+) \subseteq (yI_t)^+$ for all t, simply because $y(I_tR^+) = (yI_t)R^+$, and since $yI_t \subseteq I_{t+1}$ we have as well that $y(I_t)^+ \subseteq I_{t+1}^+$. Hence, we can consider

$$\varinjlim_t \frac{I_t^+}{I_t}$$

where the maps are induced by multiplication by y on numerators, and the direct limit is a submodule of $H_m^d(R)$. This submodule is the same as 0_H^+.

To see this, first note that because R^+ is a big Cohen-Macaulay algebra over R, $x_1, \ldots, x_d$ is a regular sequence on R^+, and so the maps in the direct limit system

$$\varinjlim_t \frac{R^+}{I_tR^+}$$

are injective, and each R^+/I_tR^+ injects into $H_m^d(R^+)$. It follows that if $v \in H_m^d(R)$ is represented by the class of $u \in R$ in R/I_t, then v maps to 0 in $H_m^d(R^+)$ if and only if $u \in I_tR^+$ if and only if $u \in I_tR^+ \cap R = I_t^+$, from which the result follows.

Moreover, we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \frac{R^+}{I_tR^+} \arrow[r, "y\cdot"] & \frac{R^+}{I_{t+1}R^+} \arrow[r] & \cdots\ H^d_m(R^+) \\
\cdots \arrow[r] & \frac{R}{I_t} \arrow[u] \arrow[r, "y\cdot"] & \frac{R}{I_{t+1}} \arrow[u] \arrow[r] & \cdots\ H^d_m(R) \arrow[u] \\
\cdots \arrow[r] & \frac{I^*_t}{I_t} \arrow[u] \arrow[r, "y\cdot"] & \frac{I^*_{t+1}}{I_t} \arrow[u] \arrow[r] & \cdots\ 0^*_{H^d_m(R)} \arrow[u] \\
\cdots \arrow[r] & \frac{I^+_t}{I_t} \arrow[u] \arrow[r, "y\cdot"] & \frac{I^+_{t+1}}{I_t} \arrow[u] \arrow[r] & \cdots\ 0^+_{H^d_m(R)} \arrow[u]
\end{tikzcd}$$

where the vertical maps, except those to the top row, are injective, and so $0^+_H \subseteq 0^*_H \subseteq H$. It also follows that

$$\frac{0^*_H}{0^+_H} = \varinjlim{}_t \frac{I^*_t}{I^+_t}$$

for every choice of system of parameters $x_1, \ldots, x_d$ for R.

When R is Cohen-Macaulay and, in particular, when R is Gorenstein, all of the horizontal maps in the commutative diagram just above are injective, as well as the vertical maps other than those to the top row.

Step 5. Reformulation of the problem in terms of $0^*_{H^d_m(R)}/0^+_{H^d_m(R)}$ *and its dual.* In consequence of the discussion above, we can assert the following:

Proposition. *The following three conditions on an excellent Gorenstein domain* (R, m, K) *of prime characteristic* $p > 0$ *of Krull dimension* d *are equivalent.*

(1) *For every system of parameters* $x_1, \ldots, x_d \in m$, *if* $I = (x_1, \ldots, x_d)R$, *then* $I^* = I^+$.

(2) *For some system of parameters* $x_1, \ldots, x_d \in m$, *if* $I_t = (x_1^t, \ldots, x_d^t)R$, *then* $I^*_t = I^+_t$ *for all* $t \geq 1$.

(3) *If* $H = H^d_m(R)$, *then* $0^*_H/0^+_H = 0$, *i.e.,* $0^*_H = 0^+_H$.

Proof. It is obvious that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). We need only show that (3) $\Rightarrow$ (1). Assume (3). If $x_1, \ldots, x_d$ is a system of parameters and I_t is defined as in (2), we may use this system to calculate

$$\frac{0^*_H}{0^+_H} = \varinjlim{}_t \frac{I^*_t}{I^+_t}.$$

The maps in the direct limit system are all injective. Hence, if $I^* \neq I^+$, we cannot have $0^*_H = 0^+_H$. This contradiction proves that (3) $\Rightarrow$ (1). $\square$

We next note the following easy consequence of Matlis duality.

Proposition. *Let (R, m, K) be a local ring, and let $E = E_R(K)$ be an injective hull of the residue class field. Let $\_^\vee$ denote the functor* $\mathrm{Hom}_R(\_, E)$.

(a) *If R is complete, there is a bijective order-reversing correspondence between submodules $N \subseteq E$ and ideals of R under which $N \subseteq E$ corresponds to* $\mathrm{Ann}_R N$ *and $J \subseteq R$ corresponds to* $\mathrm{Ann}_E J$. *In particular, if $N \subseteq E$,* $\mathrm{Ann}_E(\mathrm{Ann}_R N) = N$, *and if $J \subseteq R$, then* $\mathrm{Ann}_R(\mathrm{Ann}_E J) = J$. *$N$ corresponds to J if and only if N is the Matlis dual $(R/J)^\vee$ of R/J, in which case $N \cong E_{R/J}(K)$.*

(b) *Whether R is complete or not, if $J \subseteq R$ is any ideal,* $\mathrm{Ann}_R(\mathrm{Ann}_J E) = J$.

Proof. (a) Note that if M has ACC or DCC, we have that $\mathrm{Ann}_R(M) \subseteq \mathrm{Ann}_R(M^\vee)$, and $\mathrm{Ann}_R(M^\vee) \subseteq \mathrm{Ann}_R(M^{\vee\vee}) = \mathrm{Ann}_R M$ in turn, since $M^{\vee\vee} \cong M$. Thus, M and $M^\vee$ have the same annihilator.

There is a bijection between injections $N \hookrightarrow E$ and surjections $N^\vee \twoheadleftarrow R$ obtained by applying $\_^\vee$ (this is used in both directions). Thus, $N \hookrightarrow E$ is dual to $R/J \twoheadleftarrow R$ for some ideal of J of R that is uniquely determined by N. Since N and R/J have the same annihilator, $J = \mathrm{Ann}_R N$. The dual of R/J is evidently $\mathrm{Hom}_R(R/J, E) \cong \mathrm{Ann}_E J$.

(b) Let $\mathrm{Ann}_E J = \mathrm{Ann}_E(J\widehat{R})$, and so the annihilator of $\mathrm{Ann}_{\widehat{R}}(\mathrm{Ann}_J E) = J\widehat{R}$. It follows that the annihilator of $\mathrm{Ann}_E J$ in R is $J\widehat{R} \cap R = J$, since whR is faithfully flat over R. $\square$

For the rest of the proof of Theorem that plus closure and tight closure agree for ideals generated by a system of parameters, if R is a complete local Gorenstein domain of prime characteristic $p > 0$ of Krull dimension d, we shall write write $H = H^d_m((R)$, and we shall write J_* for $\mathrm{Ann}_R(0^*_H)$ and J_+ for $\mathrm{Ann}_R(0^+_H)$. We shall see that $J_* = \tau(R) = \tau_{\mathrm{b}}(R)$ in the Gorenstein case. Our objective is to show that $0^*_H = 0^+_H$. We make use of the following fact.

Corollary. *With notation as above, $0^*_H/0^+_H$ is the Matlis dual of J_+/J_*.*

Proof. Since R is Gorenstein, $H = E$ is an injective hull for R and we may take the Matlis dual to of a given module M to be $M^\vee = \mathrm{Hom}_R(M, H)$. We have a short exact sequence

$$0 \to 0^+_H \to 0^*_H \to 0^*_H/0^+_H \to 0$$

whose dual is

$$0 \leftarrow R/J_+ \leftarrow R/J_* \leftarrow (0^*_H/0^*_H)^\vee \leftarrow 0.$$

The kernel of the map on the left is evidently J_+/J_*, from which the result follows at once. $\square$

We next want to establish the connection between J_* and the test ideal of R. Part (d) of the result just below was also discussed in the solution of problem 4. in Problem Set #3.

Theorem. *Let R be a reduced Noetherian ring of prime characteristic $p > 0$. Let $E_R(M)$ denote an injective hull of M over R. Note that if m is a maximal ideal, $E_R(R/m) \cong E_{R_m}(R_m/MR_m)$.*

(a) $\tau(R) = \displaystyle\bigcap_{m \in \mathrm{MaxSpec}(R)} \mathrm{Ann}_R(0^{*\mathrm{fg}}_{E_R(R/m)})$.

(b) $\tau_{\mathrm{b}}(R) = \displaystyle\bigcap_{m \in \mathrm{MaxSpec}(R)} \mathrm{Ann}_R(0^{*}_{E_R(R/m)})$.

(c) *Hence, if (R, m, K) is local and $E = E_R(K)$, then $\tau(R) = \mathrm{Ann}_R(0^{*\mathrm{fg}}_E)$ and $\tau_{\mathrm{b}}(R) = \mathrm{Ann}_R(0^*_E)$.*

(d) *If R is local and approximately Gorenstein with I_t a descending sequence of m-primary irreducible ideals cofinal with the powers of m, then*

$$\tau(R) = \bigcap_t I_t :_R I_t^*.$$

(e) *If R is local, excellent, and Gorenstein, then $\tau(R) = \tau_{\mathrm{b}}(R) = \mathrm{Ann}(0^*_H)$, where $H = H^d_m(R)$.*

Proof. Part (c) is just a restatement of (a) and (b) in the local case.

In all of (a), (b), and (d), $\subseteq$ is clear. Suppose that c is in the specified intersection but that we have modules $N \subseteq M$ such that $u \in N^*_M$ and $cu \notin N$, and assume as well that these modules are finitely generated in cases (a) and (d). We are free to replace N with a submodule of M containing N and maximal with respect to not containing cu, and we ma then kill N. The image of cu is then killed by some maximal ideal m of R, and generates a module $V \cong R/m = K$ in a module M that is an essential extension of V. In case (a), M is a finitely generated submodule of $E = E_R(R/m)$, with $u \in 0^*$, and so $u \in 0^{*\mathrm{fg}}_E$, which implies that $cu = 0$, a contradiction. In case (d), M is killed by I_t for some t, and so may be viewed as an essential extension of K over the Gorenstein Artin local ring R/I_t. But then M injects into R/I_t. The image v of u is in the tight closure of 0 in R/I_t, but cv is not 0. If we represent v by an element $r \in R$, we have that $r \in I_t^*$ but that $cr \notin I_t$, a contradiction. Finally, in case (b), M embeds in E, and $u \in 0^*_E$ while $cu \neq 0$, a contradiction.

Part (e) is then immediate from the fact that in the local, excellent, Gorenstein case, $H = E$ and $0^*_E = 0^{*\mathrm{fg}}_E$. $\square$

We shall now complete the proof by showing that J_+/J_* and, hence, $0^*_H/0^+_H$, has finite length in the case of a minimal counterexample, and then applying the Theorem on p. 3 of the Lecture Notes from November 21 on killing finitely generated submodules of local cohomology by making a suitable module-finite extension. A key point is that information about J_+/J_* can be obtained by localizing at a proper prime ideal P of R, which is not directly true for $0^*_H/0^+_H$.