Math 711: Lecture of December 3, 2007

Step 6. Proof that J_+/J_* has finite length when d is minimum. We first prove that the test ideal commutes with localization for a reduced excellent Gorenstein local ring of prime characteristic $p > 0$. In order to do so, we introduce a new notion. Let (R, m, K) be a reduced excellent Gorenstein local ring of prime characteristic $p > 0$. We say that an ideal $J \subseteq R$ is *F-stable* provided that J is the annihilator of a submodule N of $H = H^d_m(R)$ that is stable under the action of F on H. We first note:

Lemma. *Let notation be as above.*

(a) *$J \subseteq R$ is F-stable if and only if for every ideal I of R generated by a system of parameters $x_1, \ldots, x_d$, $(*)$ if $Ju \subseteq I$ then $Ju^p \subseteq I^{[p]}$. Moreover, for J to be F-stable, it suffices that for a single system of parameters $x_1, \ldots, x_d$ for R, with $I_t = (x_1^t, \ldots, x_d^t)R$ we have that $Ju \subseteq I_t$ implies that $Ju^p \subseteq I_t^{[p]}$ for all t.*

(b) *If J is F-stable and P is any prime ideal of R, then JR_P is F-stable.*

(c) *If J is an F-stable ideal of R that contains a nonzerodivisor, then $\tau(R) \subseteq J$.*

Proof. (a) Each element of H is represented by the class v of $u \in R$ in $R/I_t \hookrightarrow H$ for some t. The maps in the direct limit system for H are injective, and so J kills the class of u if and only if $Ju \subseteq I_t$. Then $F(v)$ is represented by v^p in R/I_{pt}, and $I_{pt} = I_t^{[p]}$. It is immediate both that condition $(*)$ for all I_t is necessary and sufficient for J to be F-stable, and since we may choose I to be any ideal generated by a system of parameters, we must have $(*)$ for all parameter ideals.

(b) Let $h = \operatorname{height}(P)$ and choose a system of parameters $x_1, \ldots, x_d \in m$ such that $x_1, \ldots, x_h \in P$, and such that the images of $x_1, \ldots, x_h$ form a system of parameters in R_P. It suffices to check that if $u/w \in R_P$, where $u \in R$ and $w \in R - P$, and $J(u/w)^p \in (x_1, \ldots, x_h)R_P$, then $J(u/w) \in (x_1^p, \ldots, x_h^p)R_P$. From the first condition we can choose $w' \in R - P$ such that $w'Ju \subseteq (x_1, \ldots, x_h)R$, and the latter ideal is contained in $(x_1, \ldots, x_h, x_{h+1}^N, \ldots, x_d^N)R$ for all $N \geq 1$. Since J is F-stable, and $J(w'u) \subseteq (x_1, \ldots, x_h, x_{h+1}^N, \ldots, x_d^N)R$, we have that

$$J(w'u)^p \subseteq (x_1^p, \ldots, x_h^p, x_{h+1}^{pN}, \ldots, x_d^{pN})R$$

for all N. Intersecting the ideals on the right as N varies, we obtain that

$$(w')^p Ju^p \subseteq (x_1^p, \ldots, x_h^p),$$

which implies that $J(u/w)^p \in (x_1^p, \ldots, x_h^p)R_P$, as required.

(c) Let $c \in J \cap R^\circ$. Since c kills $N = \operatorname{Ann}_H J$ if $v \in N$ we have that $v^q \in N$ for all e, and so $cv^q = 0$ for all q. It follows that $v \in 0^*_H$. Hence, $N \subseteq 0^*_H$, and so $\tau(R) = \operatorname{Ann}_R 0^*_H \subseteq \operatorname{Ann}(N) = \operatorname{Ann}_R(\operatorname{Ann}_H(J)) = J$. $\square$

We shall need the following fact:

Proposition. *Let (R, m, K) be an excellent reduced equidimensional local ring of prime characteristic $p > 0$, and let $d = \dim(R)$. Let $H = H^d_m(R)$. Then 0^*_H is stable under the action of F. If R is a domain, 0^+_H is stable under the action of F.*

Proof. Let $x_1, \ldots, x_d$ be a system of parameters. The first statement follows from the fact that if $u \in I_t^*$ for some t, then $u^p \in \bigl((I_t)^{[p]}\bigr)^*$. In fact, if $u \in I^*$ then $u^p \in (I^{[p]})^*$ in complete generality. The second assertion follows from the fact that if $u \in I_t^+$, then $u^p \in \bigl(I_t^{[p]}\bigr)^+$. The corresponding fact for any ideal I in any domain R follows from the fact that the Frobenius endomorphism on R^+ sends u to u^p and IR^+ to $I^{[p]}R^+$ while stabilizing R: hence, $u^p \in I^{[p]}R^+ \cap R$. $\square$

We next note:

Lemma. *Let R be a Noetherian ring of prime characteristic $p > 0$. Let $\mathfrak{A}$ be an ideal whose radical is contained only in maximal ideals of R, and let m be one maximal ideal of R. Then $(\mathfrak{A}R_m)^*$ in R_m is the same as $\mathfrak{A}^*R_m$.*

Proof. It suffices to prove $\subseteq$. Let $m = m_1, \ldots, m_k$ be the maximal ideals of R. If any m_i is also minimal, then $\{m_i\}$ is an isolated point of $\operatorname{Spec}(R)$, and the ring is a product. Every ideal is a product, and tight closure may be calculated separately in each factor. We can reduce to studying a factor where tthere are fewer maximal ideals. Therefore, we may assume that no m_i is minimal.

Then $\mathfrak{A}$ has primary decomposition $\mathfrak{A} = \mathfrak{A}_1 \cap \cdots \cap \mathfrak{A}_k$ where $\mathfrak{A}_i$ is primary to m_i. Choose an element w of $\mathfrak{A}_2 \cap \cdots \cap \mathfrak{A}_k$ that is not in P, and not in any minimal prime of the ring.

Now suppose that $u/1 \in (IR_P)^*$ (we may clear denominators to assume the element has this form). By the Proposition on p. 2 of the Lecture Notes from September 17, we can choose $c \in R^\circ$ such that $cu^{[q]}/1 \in (\mathfrak{A}R_m)^{[q]}$ for all $q \gg 0$. Then $(*)\quad wc \in R^\circ \cap (R - m)$, and $c(wu)^q \in \mathfrak{A}^{[q]}$ for all $q \gg 0$. To see this, note that

$$\mathfrak{A}^{[q]} = (\mathfrak{A}_1 \cap \cdots \cap \mathfrak{A}_k)^{[q]} = (\mathfrak{A}_1 \cdots \mathfrak{A}_k)^{[q]}$$

(since the ideals $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ are pairwise comaximal). This becomes $\mathfrak{A}_1^{[q]} \cap \cdots \cap \mathfrak{A}_k^{[q]}$ and, since the ideal m_i is maximal, the ideal $\mathfrak{A}_i^{[q]}$ is primary to m_i. Then cu^q is in the contraction of $(\mathfrak{A}R_m)^{[q]}$ to R, and this is $\mathfrak{A}_1^{[q]}$, while $w^q \in \mathfrak{A}_i^{[q]}$ for $i > 1$. This proves $(*)$, and, hence, $wu \in I^*$ and $u \in W^{-1}I^*$. $\square$

Theorem (K. E. Smith). *Let (R, m, K) be an excellent reduced Gorenstein local ring of prime characteristic $p > 0$. Let P be a prime ideal of R. Then $\tau(R_P) = \tau(R)_P$.*

Proof. We know that both ideals are generated by nonzerodivisors. We first show that $\tau(R)_P \subseteq \tau(R_P)$. Let $c \in \tau(R)$. Let $\operatorname{height}(P) = h$ and let $x_1, \ldots, x_h$ be part of a system of parameters for R whose images in R_P give a system of parameters for R_P. Let $\mathfrak{A}_t = (x_1^t, \ldots, x_h^t)R$. By part (d) of the Theorem on p. 5 of the Lecture Notes from November 30, it suffices to show that for every t, $c(\mathfrak{A}_t R_P)^*_{R_P} \subseteq \mathfrak{A}_t R_P$. We claim that $(\mathfrak{A}_t R_P)^* = \mathfrak{A}_t^* R_P$. By Problem 2(a) of Problem Set #3, we can localize at the multiplicative system W which is the complement of the union of the minimal primes of $\mathfrak{A}$, since elements of W are nonzerodivisors on every $\mathfrak{A}^{[q]}$. In the resulting semilocal ring, the expansion of P is maximal, and we may apply the preceding Lemma to obtain that $(\mathfrak{A}_t R_P)^* = \mathfrak{A}_t^* R_P$. But then $c\mathfrak{A}_t^* \subseteq \mathfrak{A}_t$, and so $c(\mathfrak{A}_t R_P)^* = c\mathfrak{A}_t^* R_P \subseteq \mathfrak{A}_t R_P$, as required.

To prove the other direction, let d be the Krull dimension of R and let $H = H^d_m R)$. Then the annihilator of 0^*_H in R is $\tau(R)$. Hence, by the Proposition above, $\tau(R)$ is an F-stable ideal. It follows that $\tau(R)R_P$ is an F-stable ideal of R_P by part (b) of the Lemma on p. 1. It contains a nonzerodivisor, since $\tau(R)$ does. By part (c) of the Lemma on p. 1, $\tau(R_P) \subseteq \tau(R)R_P$. $\square$

We can now prove:

Lemma. *Let (R, m, K) be a complete local Gorenstein domain of Krull dimension d of prime characteristic $p > 0$ such that, for $h < d$, tight closure is the same as plus closure for ideals generated by h elements that are part of a system of parameters. Then J_+/J_* has finite length. Hence, $0^*/0^+$ has finite length.*

Proof. Since J_+/J_* is finitely generated, it suffices to prove that it becomes 0 when we localize at a prime ideal P of R strictly contained in m. Since $J_* = \tau(R)$, we have that $(J_*)_P = J_* R_P = \tau(R_P)$, by the Theorem above. Hence, it suffices to prove that every element of J_+ maps to a test element in R_P. Let $c \in J_+$. Let $h = \text{height}\,(P)$. Let $x_1, \ldots, x_d$ be a system of parameters for R such that $x_1, \ldots, x_h \in P$ and their images in R_P are a system of parameters for R_P. Then it suffices to show that

$$c\big((x_1^t, \ldots, x_h^t)R_P\big)^* \subseteq (x_1^t, \ldots, x_h^t)R_P$$

for all t. We have that

$$\big((x_1^t, \ldots, x_h^t)R_P\big)^* = (x_1^t, \ldots, x_h^t)^* R_P$$

and

$$\big((x_1^t, \ldots, x_h^t)R\big)^* = \big((x_1^t, \ldots, x_h^t)R\big)^+ \subseteq \big((x_1^t, \ldots, x_h^t, x_{h+1}^N, \ldots, x_d^N)R\big)^+$$

for all $N \geq 1$. Since $c \in J_+$, this yields

$$c\big((x_1^t, \ldots, x_h^t)R_P\big)^* \subseteq (x_1^t, \ldots, x_h^t, x_{h+1}^N, \ldots, x_d^N)R$$

for all $N \geq 1$. We may intersect the ideals on the right as N varies to obtain

$$c\big((x_1^t, \ldots, x_h^t)R_P\big)^* \subseteq (x_1^t, \ldots, x_h^t)R,$$

and localizing at P then gives the result that we require. $\square$

Step 7. The dénouement: applying the Theorem on killing local cohomology in a module-finite extension. We can now complete the proof that plus closure is the same as tight closure for parameter ideals. We know that the kernel of the map $H = H^d_m(R) \to H^d_m(R^+)$ is 0^+_H, by the Discussion on p. 2 of the Lecture Notes from November 30. Hence, we may view $M = 0^*_H/0^+_H$ as an R-submodule of $H^d_m(R^+)$. It is stable under F, since this is true for both 0^*_H and 0^+_H in $H^d_m(R)$, by the Proposition at the top of p. 2, and the map $H^d_m(R) \to H^d_m(R^+)$ commutes with the action of F. Hence, by the Theorem on p. 3 of the Lecture Notes of November 21, there is a module finite extension domain T of R^+ such that the map $H^d_m(R^+) \to H^d_m(T)$ kills m. However, R^+ does not have such an extension, unless it is an isomorphism. Hence, M must already be 0 in $H^d_m(R^+)$, which shows that $0^*_H = 0^+_H$. This completes the proof of the Theorem stated at the bottom of p. 4 of the Lecture Notes of November 21, as sketched on p. 5 of those Lecture Notes. $\square$

Characterizing tight closure using solid algebras and big Cohen-Macaulay algebras

We next want to prove the results on characterizing tight closure over complete local domains using solid algebras and big Cohen-Macaulay algebras that were stated on p. 12 of the Lecture Notes of September 7.

We recall that an R-module M over a domain R is *solid* if $\mathrm{Hom}_R(M, R) \neq 0$. That is, there is a nonzero R-linear map $\theta : M \to R$.

Nonzero finitely generated torsion-free modules are solid if and only if they are not torison modules: the quotient by the torsion submodule is finitely generated, nonzero, and torsion free. It can be embedded in a finitely generated free R-module, and one of the coordinate projections will give a nonzero map to R. However,we will be primarily interested in the case where M is an R-algebra. Solidity is much more difficult to understand in this case.

Note that if S is an R-algebra, then S is solid if and only if there is an R-linear module homomorphism $\theta : S \to R$ such that $\theta(1) \neq 0$. For if $\theta_1 : S \to R$ is an R-linear module homomorphism such that $\theta_1(s_0) \neq 0$, we can define θ to be the composition of this map with multiplication by s_0, i.e., define $\theta(s) = \theta(s_0 s)$ for all $s \in S$.

The following is a slight generalization of Problem 1 of Problem Set #1.

Proposition. *Let R be a Noetherian domain of prime characteristic $p > 0$, and let $N \subseteq M$ be R-modules. If $u \in M$ is such that $1 \otimes u \in \langle S \otimes_R N\rangle$ in $S \otimes_R M$, then $u \in N^*_M$.*

Proof. Let $\theta : S \to R$ be an R-linear map such that $\theta(1) = c \neq 0$. Then $u^q \in \langle (S \otimes N)^{[q]}\rangle$ in

$$\mathcal{F}^e_S(S \otimes_R M) \cong S \otimes_R \mathcal{F}^e_R(M),$$

and we may identify $\langle (S \otimes_R N)^{[q]}\rangle$ with $\langle S \otimes_R N^{[q]}\rangle$. Apply $\theta \otimes_R \mathbf{1}_{\mathcal{F}^e(M)}$ to obtain that $cu^q \in N^{[q]}$ in $\mathcal{F}^e_R(M)$ for all q. $\square$

Our objective is to prove a converse for finitely generated modules over complete local domains.

Theorem. *Let $(R,\, m,\, K)$ be a complete local domain of prime characteristic $p > 0$. Let $N \subseteq M$ be finitely generated R-modules, and let $u \in M$. The following conditions are equivalent.*

(a) $u \in N^*_M$.

(b) *There exists a solid R-algebra S such that $1 \otimes u \in \langle S \otimes_R N\rangle$ in $S \otimes_R M$.*

(c) *There exists a big Cohen-Macaulay R-algebra S such that $1 \otimes u \in \langle S \otimes_R N\rangle$ in $S \otimes_R M$.*

It will be some time before we can prove this. We shall actually prove that there is an R^+-algebra B that is a big Cohen-Macaulay algebra for every module-finite extension R_1 of R within R^+, and the B can be used to test all instances of tight closure in finitely generated modules over such rings R_1.

Before beginning the argument, we want to give quite a different characterization of solid modules over a complete local domain R. In the following result, there is no restriction on the characteristic.

Theorem. *Let $(R,\, m,\, K)$ be a complete local domain of Krull dimension d, and let M be any R-module. Then M is solid if and only if $H^d_m(M) \neq 0$.*

Proof. We know that R is a module-finite extension of a regular local ring $(A,\, m_A,\, K)$, and for any R-module N, $H^d_m(N) \cong H^d_{m_A}(N)$. First suppose that M is solid, and admits a nonzero map $M \to R$. The long exact sequence for local cohomology yields

$$\cdots \to H^d_{m_A}(M) \to H^d_{m_A}(J) \to 0,$$

where $J \neq (0)$ is some ideal of R, since $H^{d+1}_{m_A}$ vanishes on all A-modules (m_A is generated by d elements). Hence, it suffices to see that $H^d_{m_A}(J) \neq 0$. Since J is a torsion-free A-module, it contains a nonzero free A-submodule, A^h whose quotient is a torsion A-module (take h as large as possible). Then we have $0 \to A^h \to J \to C \to 0$ where C is killed by some element $x \in m_A - \{\{0\}$. Then we also have $J \cong xJ \subseteq A^h \to C' \to 0$, and C' is killed by x since $xA^h \subseteq xJ$. Since C' is a module over A/xA, whose maximal ideal is the radical of an ideal with $d-1$ generators, we have that $H^d_{m_A}(C') = H^d_{m_A/xA}(C') = 0$, and so the long exact sequence for local cohomology yields

$$\cdots \to H^d_{m_A}(J) \to H^d_{m_A}(A^h) \to 0.$$

Since $H^d_{m_A}(A^h) \cong H^d_{m_A}(A)^{\oplus h}$ and $H^d_{m_A}(A) \cong E_A(A/m_A) \neq 0$, we have that $H^d_{m_A}(J) \neq 0$ and, hence, $H^d_m(M) \neq 0$, as claimed.

Now suppose that $H^d_m(M) \neq 0$. Let $x_1,\, \ldots\,, x_d$ be a system of parameters for A. Let $I_t = (x_1^t,\, \ldots\,, x_d^t)A$. Let $E = H^d_{m_A}(A)$, which is also an injective hull for A/m_A over

A. Then $H^d_m(R) = H^d_{m_A}(M) = \varinjlim_t M/I_tM \cong M \otimes_A \varinjlim_t A/I_t \cong M \otimes_A E \neq 0$. Now $\mathrm{Hom}_A(\_\,, E)$ is a faithfully exact functor on A-modules. (To see that it does not vanish on $N \neq 0$, choose a nonzero element $v \in N$. The $Av \cong A/\mathfrak{A}$ for some proper ideal $\mathfrak{A}$, which yields a map $Av \twoheadrightarrow K \hookrightarrow E$. This nonzero map from Av to E extends to N because E is injective over A.) Hence,

$$\mathrm{Hom}_A(M \otimes_A E,\, E) \neq 0.$$

By the adjointness of $\otimes$ and Hom, we have that

$$\mathrm{Hom}_A\big(M,\, \mathrm{Hom}_A(E,\, E)\big) \neq 0,$$

and since A is complete, we have that $\mathrm{Hom}_A(E,\, E) \cong A$. Thus, $\mathrm{Hom}_A(M,\, A) \neq 0$, i.e., M is solid over A.

We have a nonzero A-linear map $\eta : M \to A$. Exactly as in the argument that begins near the bottom of p. 1 in the Lecture Notes from November 14, we have an induced map $\eta_* : \mathrm{Hom}_A(R,\, M) \to \mathrm{Hom}_A(R, A)$, and $\mathrm{Hom}_A(R,\, A)$ is a torsion-free R-module of rank one and, hence, isomorphic with a nonzero ideal $J \subseteq R$. As in the Lecture Notes from November 14, we consequenctly have a composite R-module map

$$M \xrightarrow{\mu} \mathrm{Hom}_A(R,\, M) \xrightarrow{\eta_*} \mathrm{Hom}_A(R,\, A) \xrightarrow{\cong} J \xrightarrow{\subseteq} R$$

where the first map μ is the map that takes $u \in M$ to the map $f_u : R \to M$ such that $f_u(r) = ru$ for all $r \in R$. Call the composite map θ. Let $v \in M$ be such that $\eta(v) \neq 0$. Then $\theta(v) \in R$ is the image under an injection of a map $g : R \to A$ whose value on 1 is $\eta(v) \neq 0$, and so $\theta(v) \neq 0$ and, hence, $\theta \neq 0$. $\square$

Remark. The argument in the last paragraph shows that, in general, if R is a domain that is a module-finite extension of A and M is an R-module that is solid when viewed as an A-module, then M is also solid as an R-module.