**Math 711: Lecture of December 5, 2007**

From the local cohomology criterion for solidity we obtain:

**Corollary.** *A big Cohen-Macaulay algebra (or module) $B$ over a complete local domain $R$ is solid.*

*Proof.* Let $d = \dim(R)$ and let $x_1, \ldots, x_d$ be a system of parameters for $R$. This is a regular sequnece on $B$, and so the maps in the direct limit system

$$B/(x_1, \ldots, x_d)B \to \cdots \to B/(x_1^t, \ldots, x_d^t)B \to \cdots$$

are injective. Since $0 \neq B/(x_1, \ldots, x_d)B \hookrightarrow H^d_{(\underline{x})}(B) = H^d_m(B)$, $B$ is solid. $\square$

Our next objective is to prove:

**Theorem.** *Let $R$ be a complete local domain. Then there exists an $R^+$-algebra $B$ such that for every ring $R_1$ with $R \subseteq R_1 \subseteq R^+$ such that $R_1$ is module-finite over $R$ the following two conditions hold:*

(1) *$B$ is a big Cohen-Macaulay algebra for $R_1$.*

(2) *For every pair of finitely generated $R_1$-modules $N \subseteq M$ and $u \in N^*_M$, $1 \otimes u \in \langle B \otimes_R N\rangle$ in $B \otimes_R M$.*

The proof will take a considerable effort. The basic idea is to construct an algebra $B$ with the required properties by introducing many indeterminates and killing the relations we need to hold. The difficulty will be to prove that in the resulting algebra, we have that $mB \neq B$.

## Forcing algebras

Let $T$ ba ring, $u$ an $h \times 1$ column vector over $T$, and let $\alpha$ be an $h \times k$ matrix over $T$. Let $Z_1, \ldots, Z_k$ be indeterminates over $T$ and let $I$ be the ideal generated by the entries of the matrix

$$u - \alpha \begin{pmatrix} Z_1 \\ \vdots \\ Z_k \end{pmatrix}$$

By the *forcing algebra*, which we denote $\mathfrak{Force}_\sigma(T)$, of the pair $\sigma = (u, \alpha)$ over $T$ we mean the $T$-algebra $T[Z_1, \ldots, Z_k]/I$. In this algebra, we have "forced" $u$ to be a linear combination (the coefficients are the images of the $Z_i$) of the columns of $\alpha$. If $M$ is the cokernel of the matrix $\alpha$, we have that $1 \otimes u = 0$ in $\mathfrak{Force}_\sigma(T) \otimes_T M$. Given any other

$T$-algebra $T'$ such that $1 \otimes u = 0$ in $T' \otimes M$ (equivalently, such that the image of $u$ is a $T'$-linear combination of the images of the columns of $\alpha$), we have a $T$-homomorphism $\mathfrak{F}\mathtt{orce}_\sigma(T) \to T'$ that sends the $Z_i$ to the corresponding coeffiicients in $T'$ used to express the image of $u$ as a linear combination of the images of the columns of $\alpha$. We shall say that $\mathfrak{F}\mathtt{orce}_\sigma(T)$ is obtained from $T$ by *forcing* $\sigma$.

It will be technically convenient to allow the matrix $\alpha$ to have size $h \times 0$, i.e., to have no columns. In this case, the forcing algebra is formed by killing the entries of $u$. (Typically, we are forcing $u$ to be in the span of the columns of $\alpha$. When $k = 0$, the span of the empty set is the 0 submodule in $T^h$.)

Now suppose that we are given a set $\Sigma$ of pairs of the form $(u, \alpha)$ where $u$ is a column vector over $T$ and $\alpha$ is a matrix over $T$ whose columns have the same size as $u$. We call the set $\Sigma$ *forcing data* for $T$. The size of $u$ and of the matrix may vary. By the *forcing algebra* $\mathfrak{F}\mathtt{orce}_\Sigma(T)$ we mean the coproduct of the forcing algebras $\mathfrak{F}\mathtt{orce}_\sigma(T)$ as $\sigma$ varies in $T$. One way of constructing this coproduct is to adjoin to $T$ one set of appropriately many indeterminates for every $\sigma \in \Sigma$, all mutually algebraically independent over $T$, and then impose for every $\sigma$ the same relations needed to form $\mathfrak{F}\mathtt{orce}_\sigma(T)$. If $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ one may think of this algebra as

$$\bigotimes_{i=1}^{n} \mathfrak{F}\mathtt{orce}_{\sigma_i}(T),$$

where the tensor product is taken over $T$. When $\Sigma$ is infinite, one may think of $\mathfrak{F}\mathtt{orce}_\Sigma(T)$ as the direct limit of all the forcing algebras for the finite subsets $\Sigma_0$ of $\Sigma$.

If $\Sigma$ is forcing data for $T$ and $h : T \to T'$, we may take the image of $\Sigma$ to get forcing data over $T'$: one is simply applying the homomorphism $h$ to every entry of every column and every matrix. We write $h(\Sigma)$ for the image of $\Sigma$ under $h$. Then

$$\mathfrak{F}\mathtt{orce}_{h(\Sigma)}(T') \cong T' \otimes_T \mathfrak{F}\mathtt{orce}_\Sigma(T).$$

We refer to the process of formation of $\mathfrak{F}\mathtt{orce}_{h(\Sigma)}(T')$ as *postponed* forcing: we have, in fact, postponed the formation of the forcing algebra until after mapping to $T'$.

In discussing forcing algebras we make the following slight generalization of the notations. Suppose that $\Sigma$ is a set of forcing data over $S$ and $g : S \to T$ is a homomorphism. We shall also write $\mathfrak{F}\mathtt{orce}_\Sigma(T)$ for $\mathfrak{F}\mathtt{orce}_{h(\Sigma)}(T)$. Thus, our notation will not distinguish between forcing and postponed forcing.

Note that if we partition forcing data $\Sigma$ for $T$ into two sets, we can form a forcing algebra for the first subset, and then form the forcing algebra for the image of the second subset. We *postponed* the forcing process for the second subset. But the algebra obtained in the two step process is isomorphic to $\mathfrak{F}\mathtt{orce}_\Sigma(T)$.

One can also think of the formation of $\mathfrak{F}\mathtt{orce}_\Sigma(T)$ as an infinite process. Well order the set $\Sigma$. Now perform forcing for one $(u, \alpha)$ at a time. Take direct limits at limit ordinals. By transfinite recursion, one reaches the same forcing algebra $\mathfrak{F}\mathtt{orce}_\Sigma(T)$ that one gets by doing all the forcing in one step.

## Algebra modifications

Let $g : S \to T$ be a ring homomorphism (which may well be the identity map) and let $\Gamma$ be a a set whose elements are finite sequences of elements of $S$. We assume that if a sequence is in $\Gamma$, then each initial segment of it is also in $\Gamma$. Let $\Sigma_{\Gamma,\,g}$ denote the set of pairs $\bigl(u,\,(x_1\ \ldots\ x_k)\bigr)$ such that $x_1,\,\ldots\,,x_{k+1}$ is a sequence in $\Gamma$, $u \in T$, and $x_{k+1}u \in (x_1,\,\ldots\,,x_k)T$. Thus, if $x_1,\,\ldots\,,x_{k+1}$ were a regular sequence on $T$, we would have that $u \in (x_1,\,\ldots\,,x_k)T$. Let $\Sigma$ be a subset of $\Sigma_{\Gamma,g}$. We call $\Sigma$ *modification data* for $T$ over $\Gamma$. We refer to $\mathfrak{F}\mathtt{orce}_\Sigma(T)$ as a *multiple algebra modification* of $T$ over $\Gamma$. We write $\mathsf{A}lgmod_{\Gamma,g}(T)$ for the forcing algebra $\mathfrak{F}\mathtt{orce}_{\Sigma_{\Gamma,g}}(T)$ and refer to it as the *total algebra modification* of $T$ over $\Gamma$. If $\sigma$ is one element of $\Sigma_{\Gamma,g}$ we refer to $\mathfrak{F}\mathtt{orce}_\sigma(T)$ as an *algebra modification* of $T$. For emphasis, we also refer to it as a *simple* algebra modification of $T$. We shall use iterated multiple algebra modifications to construct $T$-algebras on which the specified sequences in $\Gamma$ become regular sequences.

Note that if $k = 0$ and $x_1u = 0$ in $T$, then we get an algebra modification $\mathfrak{F}\mathtt{orce}_\sigma(T)$ in which $\sigma$ is a pair consisting of $u$ and a $1 \times 0$ matrix: this algebra modification is simply $T/uT$.

If we have a homomorphism $h : T \to T'$, and $\sigma = \bigl(u,\,(x_1\ \ldots\ x_k)\bigr)$, we write $h(\sigma)$ for $\bigl(h(u),\,(x_1\ \ldots,\ x_k)\bigr)$. Note that if $x_{k+1}u \in (x_1,\,\ldots\,,x_k)T$, then

$$x_{k+1}h(u) \in (x_1,\,\ldots\,,x_k)T'.$$

Thus,

$$h(\Sigma_{\Gamma,g}) \subseteq \Sigma_{\Gamma,h\circ g)}.$$

With this notation, if $\Sigma$ is algebra modification data for $T$ over $\Gamma$, then $h(\Sigma)$ is modification data for $T'$ over $\Gamma$.

Algebra modifications are forcing algebras. As in the general case of a forcing algebra, we may talk about postponed modifications.

## The construction of big Cohen-Macaulay algebras that capture tight closure

Let $(R,\,m,\,K)$ be a complete local domain. Let $\Gamma$ consist of all sequences in $R^+$ that are part of a system of parameters in some ring $R_1$ with $R \subseteq R_1 \subseteq R^+$ such that $R_1$ is module-finite over $R$. Let $\Sigma$ be the set of all pairs $(u,\,\alpha)$ consisting, for some ring $R_1$ as above, of an $h \times 1$ column vector over $R_1$ and an $h \times k$ matrix over a ring $R_1$ such that $u$ is in the tight closure over $R_1$ of the column space of $\alpha$. Note that we know that whether this condition holds is unaffected by replacing $R_1$ by a larger ring $R_2$ such that $R_1 \subseteq R_2 \subseteq R^+$ with $R_2$ module-finite over $R$.

Let $B_0 = \mathfrak{F}\mathtt{orce}_\Sigma(R^+)$. If $B_n$ has been defined for $i \geq 0$, let $B_{n+1}$ be the total algebra modification of $\mathbb{A}lgmod_\Gamma(B_n)$ of $B_n$ over $\Gamma$. Then we have a direct limit system

$$B_0 \to B_1 \to \cdots \to B_n \to B_{n+1} \to \cdots .$$

Let $B = \varinjlim_n B_n$. We shall prove that $B$ is the required big Cohen-Macaulay algebra.

Much of this is obvious. Suppose that $N \subseteq M$ are finitely generated $R_1$-modules and $u \in N_M^*$ over $R_1$. Choose a finite presentation for $M/N$ over $R_1$, so that $MN/$ is the cokernel of an $h \times k$ matrix $\alpha$o over $R_1$. The image of $v$ in $M/N$ is represented by an element $u \in R_1^k$. Then $u$ is in the tight closure of the column space of $\alpha$ in $R_1^h$, and it follows from the definition of $B_0$ that $u$ is a linear combination of the columns of $\alpha$ in $B_0$ and, hence, in $B$.

It is likewise easy to see that if $x_1, \ldots, x_d$ is a system of parameters in $R_1$ then it is a regular sequence on $B$. Suppose that we have a relation

$$x_{k+1}b_{k+1} = \sum_{i=1l}^{k} x_i b_i$$

on $B$. Because $B$ is the direct limit of the $B_n$, we can find $n_0$ such that $B_{n_0}$ contains elements $\beta_i$ that map to the $b_i$. The corresponding relation may not hold in $B_{n_0}$, but since it holds in $B$ it will hold when we map to $B_n$ for some $n \geq n_0$. Thus, we may assume that we have $\beta_1, \ldots, \beta_{k+1} \in B_n$ such that $x_{k+1}\beta_{k+1} = \sum_{i=1}^k x_i\beta_i$ in $B_n$ and every $\beta_i$ maps to $b_i$ when we map $B_n \to B$. By the construction of $B_{n+1}$, we have that the image of $\beta_{k+1}$ is in $(x_1, \ldots, x_k)B_{n+1}$. We can then map to $B$ to obtain that $b_{k+1} \in (x_1, \ldots, x_k)B$.

There remains only one thing to check: that $mB \neq B$. This is the most difficult point in the proof. This is equivalent to the condition that for some (equivalently, every) system of parameters $x_1, \ldots, x_d$ in every $R_1$, we have that $(x_1, \ldots, x_d)B \neq B$. To see this, observe that if $IB = B$, then $I^2B = IB = B$, and multiplying by $I$ repeatedly yields by induction that $I^tB = B$ for every $t \geq 1$. If $m_1$ is the maximal ideal of $R_1$, then $mR_1 \subseteq m_1$, which is contained in a power of $mR_1$. Hence, $mB = B$ if and only if $m_1B = B$. Likewise, if $x_1, \ldots, x_d$ is a system of parameters for $R_1$ generating an ideal $I$, for some $t$ we have $m_1^t \subseteq I \subseteq m_1$, and so $m_1B = B$ if and only if $IB = B$.

If $(x_1, \ldots, x_d)B = B$, then we have that

$$1 = x_1b_1 + \cdots + x_db_d$$

for some finite set of elements of $B$.

We shall show that if this happens, it happens for some algebra obtained from some choice of $R_1$ by a finite sequence of forcing algebra extensions: the first extension is the forcing algebra for a single pair $(u, \alpha)$ such that $u$ is in the tight closure of the column space of $\alpha$ over $R_1$. The extensions after that are simple algebra modifications with respect to sequences each of which is part of a system of parameters for $R_1$.

Eventually, we shall have to work even harder to make the problem more “finite”: specifically, we aim to replace these algebras by finitely generated submodules. We then use characteristic $p$ methods to get a contradiction. However, the first step is to get to the case where we are only performing the forcing procedure finitely many times. With suffiicient thought about the situation, that one can do this is almost obvious, but it is not quite so easy to give the argument formally..

### Descent of forcing algebras

Let $g : S \to T$ be a ring homomorphism. Let $\Lambda$ be a directed poset and let $\{S_\lambda\}_{\lambda\in\Lambda}$ be a directed family of subrings of $S$ indexed by $\Lambda$ whose union is $S$. Let $\{T_\lambda\}_{\lambda\in\Lambda}$ be a directed family of rings such that $\varinjlim_\lambda T_\lambda = T$, and suppose that for every $\lambda \in \Lambda$ we have a homomorphism $g_\lambda : S_\lambda \to T_\lambda$ such that if $\lambda \le \mu$ the diagram

$$\begin{tikzcd} S \arrow[r, "g"] & T \\ S_\mu \arrow[u] \arrow[r, "g_\mu"] & T_\mu \arrow[u] \\ S_\lambda \arrow[u] \arrow[r, "g_\lambda"] & T_\mu \arrow[u] \end{tikzcd}$$

commutes. Thus, $g : S \to T$ is the direct limit of the maps $S_\lambda \to T_\lambda$.

**Proposition.** *Let notation be as above.*

(a) *Let $\Sigma$ be forcing data over $S$. Let $\Sigma_\lambda$ denote a subset of $\Sigma$ such that all entries occurring are in $S_\lambda$. Suppose also that if $\lambda \le \mu$ then $\Sigma_\lambda \subseteq \Sigma_\mu$ and that the union of the sets $\Sigma_\lambda$ is $\Sigma$. Then $\mathfrak{F}\mathtt{orce}_\Sigma(T)$ is a direct limit of rings each of which is obtained from some $T_\lambda$ by a finite sequence of such extensions of $T_\lambda$ each of which is obtained from its immediate predecessor by forcing one element of $\Sigma_\lambda$.*

(b) *Let $\Gamma$ be a family of finite sequences in $S$ closed under taking initial segments, and $\Sigma \subseteq \Sigma_{\Gamma,g}(T)$. Define $\Sigma_\lambda$ to consist of all elements of $\Sigma$ whose entries are in $T_\lambda$, whose corresponding sequence is in $S_\lambda$, and such that if the sequence is $x_1, \ldots, x_{k+1}$ and the element is $(u, \big(x_1 \ \ldots \ x_k\big))$, then $x_{k+1}u \in (x_1, \ldots, x_k)T_\lambda$. Then the algebra modification $\mathfrak{F}\mathtt{orce}_\Sigma(T)$ is a direct limit of rings, each of which is obtained from $T_\lambda$ by a simple algebra modification over a subset of $\Gamma$.*

*Proof.* (a) Let $\mathcal{L}$ denote the poset each of whose elements is a pair $(\lambda, \Phi)$ where $\Phi$ is a finite subset of $\Sigma_\lambda$. The partial ordering is defined by the condition that $(\lambda, \Phi) \le (\mu, \Psi)$ precisely if $\lambda \le \mu$ and $\Phi \subseteq \Psi$. There is an obvious map

$$\mathfrak{F}\mathtt{orce}_\Phi(T_\lambda) \to \mathfrak{F}\mathtt{orce}_\Psi(T_\mu).$$

We claim that $\mathfrak{Force}_\Sigma(T)$ is the direct limit over $\mathcal{L}$ of the algebras $\mathfrak{Force}_\Phi(T\lambda)$ with $(\lambda,\,\Phi) \in \mathcal{L}$. If we fix any finite set $\Phi$ if $\Sigma$, it is contained in $\Sigma_\lambda$ for all sufficiently large $\lambda$, and the direct limit of the $\mathfrak{Force}_\Phi(T_\lambda)$ is evidently $\mathfrak{Force}_\Phi(T)$. The result follows from the fact that the direct limit over $\Phi$ of the rings $\mathfrak{Force}_\Phi(T)$ is $\mathfrak{Force}_\Sigma(T)$.

(b) Note that a relation involving a sequence in $\Gamma$ that holds in one $T_\lambda$ continues to hold in $T_\mu$ for all $\mu \geq \lambda$. Also note that if $\sigma \in \Sigma$, then for all sufficiently large $\lambda$ then entries will be in $T_\lambda$, the elements of the corresponding string $(x_1,\,\ldots\,,x_{k+1})$ will be in $\lambda$, and since one has $x_{k+1}u \in (x_1,\,\ldots\,,x_k)T$, this will also hold with $T$ replaced by $T_\lambda$ for all sufficiently large $\lambda$. From this it is clear that $\Sigma_\lambda \leq \Sigma_\mu$ if $\lambda \leq \mu$ and that the union of the $\Sigma_\lambda$ is $\Sigma$. Moreover, each time we force $T_\lambda$ with respect to an element of $\Sigma_\lambda$ we are performing a simple algebra modification. The result is now immediate from part (a). $\square$

**Lemma.** *Let $(R,\,m,\,K)$ be a complete local domain of prime characteristic $p > 0$, and let $B$ be the $R$-algbera constructed at the bottom of p. 3 and the top of p. 4. Then $B$ is a direct limit of rings of that are obtained as follows: start with a module-finite extension $R_1$ of $R$ within $R^+$, force finitely many elements $\sigma = (u,\,\alpha)$ where $u$ is in the tight closure of the column space of $\alpha$ over $R_1$, and then perform finitely many simple algebra modifications with respect to parts of systems of parameters in $R_1$.*

*Proof.* We freely use the notations from the bottome of p. 3 and top of p. 4 where $B$ is defined. We apply part (a) of the preceding Proposition, taking $S = T = R^+$. Both are the directed union of rings of the form $R_1$. It follows that $B_0$ is a direct limit of rings that are obtained from a module-finite extension $R_1$ of $R$ within $R^+$ by forcing finitely many elements $\sigma = (u,\,\alpha)$ where $u$ is in the tight closure of the column space of $\alpha$ over $R_1$. By induction on $n$ and part (b) of the preceding Proposition, every $B_n$ is a direct limit of rings of the form described in the statement of the Lemma. This follows for $B$ as well, since $B$ is the direct limit of the $B_n$.

Since every $R_1$ is again a complete local domain, to complete the proof of the Theorem stated on p. 1, we might as well change notation and repalce $R$ by $R_1$. Now, if one has a finite set of instances of tight closure over $R$, say $u_i$ is in the tight closure of the column space of $\alpha_i$ for $1 \leq i \leq t$, then the direct sum of the $u_i$ is in the tight closure of the column space of the direct sum of the matrices $\alpha_i$, $1 \leq i \leq t$. Moreover, the forcing algebra for these direct sums is the coproduct of the $t$ individual forcing algebras for the individual instances of tight closure. While it is not actually necessary to make this reduction, it does simplify the issue a bit. The problem that remains to complete the proof of the Theorem on p. 1 is to establish the following:

> *Let $R$ be a complete local domain of prime characteristic $p > 0$. Let $x_1,\,\ldots\,,x_d$ be a system of parameters for $R$. Let $T$ be a ring obtained from $R$ by first forcing one element $\sigma = (u,\,\alpha)$ such that $u$ is in the tight closure of the column space of $\alpha$ over $R$, and then performing finitely many simple algebra modifications with respect to parts of systems of parameters in $R$. Then $1 \notin (x_1,\,\ldots\,,x_d)T$.*

The next step in the argument will involve replacing the sequence of forcing algebras by a sequence of finitely generated $R$-modules.