Math 711: Lecture of December 10, 2007

We have defined an element of the homology or cohomology of a complex of finitely generated modules over a Noetherian ring R of prime characteristic $p > 0$ to be *phantom* if it is represented by a cycle (or cocycle) that is in the tight closure of the boundaries (or coboundaries) in the ambient module of the complex. We say that the homology or cohomology of a complex of finitely generated modules over R is *phantom* if every element is phantom. A left complex is called *phantom acyclic* if all of its homology is phantom in positive degree.

Discussion: behavior of modules of boundaries under base change. Note the following fact: if

$$G' \xrightarrow{d} G$$

is any R-linear map (in the application, this will be part of a complex), S is any R-algebra, and B denotes the image of d in G, then the image of

$$S \otimes_R G' \xrightarrow{\mathbf{1}_S \otimes d} S \otimes_R G$$

is the same as the image $\langle S \otimes_R B \rangle$ in $S \otimes_R G$. Note that we have a surjection $G' \twoheadrightarrow B$ and an injection $B \hookrightarrow G$, whence

$$S \otimes_R G' \to S \otimes_R G$$

factors as the composition of $S \otimes_R G' \twoheadrightarrow S \otimes_R B$ (this is a surjection by the right exactness of $S \otimes_R \_\!\_$) with $S \otimes_R B \to S \otimes_R G$.

From these comments, we obtain the following:

Proposition. *Let $R \to S$ be a homomorphism of Noetherian rings of prime characteristic $p > 0$ for which persistence[1] of tight closure holds. Let*

$$G' \to G \to G''$$

be part of a complex, and suppose that η is a phantom element of the homology H at G. Then the image of η in the homology H' of

$$S \otimes_R G' \to S \otimes_R G \to S \otimes_R G''$$

is a phantom element .

Hence, if S is weakly F-regular, the image of η in H' is 0.

[1] See the Theorem on p. 3 and Corollary on p. 4 of the Lecture Notes from November 7.

Proof. Let $z \in G$ represent η, and let B be the image of G' in G. Then $z \in B^*$, and $1 \otimes z$ represents the image of η in H'. By the persistence of tight closure for R to S, $1 \otimes z$ is in the tight closure over S of $\langle S \otimes_R B\rangle$ in $S \otimes_R G$, and by the discussion above, this is the same as the image of $S \otimes_R G'$. The final statement then follows because we can conclude, when the ring is weakly F-regular, that an element in tight closure of the module of boundaries is a boundary. $\square$

There are many circumstances in which one can prove that complexes have phantom homology, and these lead to remarkable results that are difficult to prove by other methods. We shall pursue this theme in seminar next semester. For the moment, we shall only give one result of this type. However, the result we give is already a powerful tool, with numerous applications.

Theorem (Vanishing Theorem for Maps of Tor). *Let $A \to R \to S$ be Noetherian rings of prime characteristic $p > 0$, where A is a regular domain, R is module-finite and torsion-free over A, and S is regular or weakly F-regular and locally excellent.*[2] *Then for every A-module M, the map*

$$\mathrm{Tor}_i^A(M,\, R) \to \mathrm{Tor}_i^A(M,\, S)$$

is 0 for all $i \geq 1$.

This result is also known in equal characteristic 0 when S is regular, but the proof is by reduction to characteristic $p > 0$. It is an open question in mixed characteristic, even in the case where S is the residue class field of R! We discuss this further below.

Proof of the Vanishing Theorem for Maps of Tor. Since M is a direct limit of finitely generated A-modules and Tor commutes with direct limit, it suffices to consider the case where M is finitely generated. If some

$$\mathrm{Tor}_i^A(M,\, R) \to \mathrm{Tor}_i^A(M,\, S)$$

has a nonzero element η in the image, this remains true when we localize at a maximal ideal of S that contains the annihilator of η and complete. Thus, we may assume that $(S,\, \mathcal{M})$ is a complete weakly F-regular local ring, and then persistence of tight closure holds for the map $R \to S$. Let m be the contraction of $\mathcal{M}$ to A. Then we may replace A, M, and R by their localizations at m, and so we may assume that $(A,\, m,\, K)$ is regular local. Let $P_\bullet$ be a finite free resolution of M by finitely generated free A-modules.

To complete the proof, we shall show that the complex $R \otimes_A P_\bullet$ is phantom acyclic over R, i.e., all of its homology modules in degree $i \geq 1$ are phantom. These homology modules are precisely the modules $\mathrm{Tor}_i^A(M,\, R)$. Hence, for $i \geq 1$, the image of this homology in the homology of

$$S \otimes_R (R \otimes_A P_\bullet) \cong S \otimes_A P_\bullet$$

[2] The result holds when the completions of the local rings of R are weakly F-regular, which is a consequence of either of the stated hypotheses.

is 0, by the Proposition at the bottom of p 1. But the i th homology module of the latter complex is $\mathrm{Tor}_i^A(M,\,S)$, as required.

It remains to show that $R\otimes_A P_\bullet$ is phantom acyclic. Let h be the torsion-free rank of R over A, and let $G \subseteq R$ be a free A-module of rank h. Thus, $G \cong A^h$ as an A-module. Then R/G is a torsion A-module, and we can choose $c \in A^\circ$ such that $cR \subseteq G$. Let $z \in R \otimes_A P_i$ represent a cycle for some $i \geq 1$. Let B_i be the image of $R \otimes P_{i+1}$ in $R \otimes_A P_i$. We shall show that $cz^q \in B_i^{[q]}$ in $\mathcal{F}_R^e(R\otimes_A P_i)$ for all q, which will conclude the proof.

It will suffice to show that, for all e, c kills the homology of $\mathcal{F}_R^e(R\otimes_A P_\bullet)$ in degree $i \geq 1$, since z^q is an element of the homology, and the module of boundaries in $\mathcal{F}_R^e(R\otimes_A P_i)$ is, by another application of the Proposition at the bottom of p. 1, the image of $\mathcal{F}_R^e(B_i)$, which is $B_i^{[q]}$.

By (11) on p. 3 of the Lecture Notes from September 12, we may identify

$$\mathcal{F}_R^e(R\otimes_A P_\bullet) \cong R \otimes_A \mathcal{F}_A^e(P_\bullet).$$

By the flatness of the Frobenius endomorphism of A, the complex $\mathcal{F}_A^e(P_\bullet)$ is acyclic. Since G is A-free, we have that $G\otimes_A \mathcal{F}_A^e(P_\bullet)$ is acyclic, and this is a subcomplex of $R\otimes_A \mathcal{F}_A^e(P_\bullet)$. Because $cR \subseteq G$, we have that c multiplies $R\otimes_A \mathcal{F}_A^e(P_\bullet)$ into the acyclic subcomplex $G\otimes_A \mathcal{F}_A^e(P_\bullet)$. If y is a cycle in $R\otimes_A \mathcal{F}_A^e(P_\bullet)$ in degree $i \geq 1$, then cy is a cycle in $G\otimes_A \mathcal{F}_A^e(P_\bullet)$ in degree i, and consequently is a boundary in $G\otimes_A \mathcal{F}_A^e(P_\bullet)$. But then it is a boundary in the larger complex $R\otimes_A \mathcal{F}_A^e(P_\bullet)$ as well. $\square$

We conclude by giving two consequences of the Vanishing Theorem for Maps of Tor. Both are known in equal characteristic and are open questions in mixed characteristic. Both can be proved by other means in the equal characteristic case, but it is striking that they are consequences of a single theorem.

Theorem. *The Vanishing Theorem for Maps of Tor implies, if it holds in a given characteristic, that direct summands of regular rings are Cohen-Macaulay in that characteristic.*

Before we can give the proof, we need a Lemma that will permit a reduction to the complete local case.

Lemma. *Let R and S be Noetherian rings.*

(a) *Let S be a regular ring and let J be any ideal of S. The the J-adic completion T of S is regular.*

(b) *Let $R \to S$ be a homomorphism and suppose that R is a direct summand of S. Then for every ideal I of R, the I-adic completion of R is a direct summand of the IS-adic completion of S.*

Proof. (a) Every maximal ideal $\mathcal{M}$ of T must contain the image of J, since if $u \in JT$ is not in $\mathcal{M}$, there exists a non-unit $v \in \mathcal{M}$ such that $tu+v=1$. But $v = 1-tu$ *is* invertible, since

$$1 + tu + \cdots + t^n u^n + \cdots$$

is an inverse. Hence, every maximal ideal T corresponds to a maximal ideal $\mathcal{Q}$ of S that contains J. Note that $S \to T$ is flat, and $\mathcal{Q}$ expands to $\mathcal{M}$. Hence, $S_{\mathcal{Q}} \to T_{\mathcal{M}}$ is faithfully flat, and the closed fiber is a field. It follows that the image of a regular system of parameters for $S_{\mathcal{Q}}$ is a regular system of parameters for $T_{\mathcal{M}}$.

(b) Let $f_1, \ldots, f_h \in I$ generate I. Let $X_1, \ldots, X_h$ be formal power series indeterminates over both rings. Let $\theta : S \to R$ be an R-linear retraction, Then $R[[X_1, \ldots, X_h]]$ is a direct sumnand of $S[[X_1, \ldots, X_h]]$: we define a retraction $\widetilde{\theta}$ that extends θ by letting θ act on every coefficient of a given power series. Let $\mathfrak{A}$ be the ideal of $R[[x_1, \ldots, x_h]]$ generated by the elements $X_i - f_i$, $1 \le i \le h$. Then there is an induced retraction

$$S[[x_1, \ldots, x_h]]/\mathfrak{A}S[[X_1, \ldots, X_h]] \to R[[X_1, \ldots, X_h]]/\mathfrak{A}.$$

The former may be identified with the IS-adic completion of S, and the latter with the I-adic completion of R. $\square$

Proof of the Theorem. Let R be a Cohen-Macaulay ring that is a direct summand of the regular ring S. The issue is local on R, and so we may replace R by its localization at a prime ideal P, and S by S_P. Therefore, we may assume that (R, m, K) is local. Second, we may replace R by its completion $\widehat{R}$ and S by its completion with respect to mS. The regularity of S and the direct summand property are preserved, by the Lemma just above.

Hence, we may assume without loss of generality that (R, m, K) is complete local, and then we may represent it as module-finite over a regular local ring A with system of parameters $x_1, \ldots, x_d$. Let $M = A/(x_1, \ldots, x_d)$. Then the maps

$$f_i : \mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$$

vanish for $i \ge 1$. Since $S = R \oplus W$ over R (over A is enough), the maps f_i are injective. Hence,

$$\mathrm{Tor}_i^A\bigl(A/(x_1, \ldots, x_d), R\bigr) = 0, \quad 1 \le i \le d.$$

This means that the Koszul homology $H_i(x_1, \ldots, x_d; R) = 0$ for $i \ge 1$, and, by the self-duality of the Koszul complex, also implies that

$$\mathrm{Ext}_A^i\bigl(A(x_1, \ldots, x_d)A, R\bigr) = 0, \quad 0 \le i < d.$$

It follows that the depth of R on $(x_1, \ldots, x_d)A$ is $d = \dim(A) = \dim(R)$. Hence R is Cohen-Macaulay. $\square$

Theorem. *The Vanishing Theorem for Maps of Tor, if it holds in a given characteristic, implies that the direct summand conjecture holds in that characteristic, i.e., that regular rings are direct summands of their module-finite extensions in that characteristic.*

Proof. Let $A \hookrightarrow R$ be module-finite. We want to show that this map splits. By part (a) of the Theorem on p. 3 of the Lecture Notes from September 24, it suffices to show this when (A, m, K) is local. By part (b), we may reduce to the case where A is complete. We may kill a minimal prime $\mathfrak{p}$ of R disjoint from A°. If $A \to R/\mathfrak{p}$ has a splitting θ, the composite map $R \twoheadrightarrow R/\mathfrak{p} \xrightarrow{\theta} A$ splits $A \hookrightarrow R$. Hence, we may also assume that R is a domain module-finite over A. Then R is local. Let L be the residue field of R. Let $x_1, \ldots, x_d$ be a regular sequence of parameters for A. The the image of 1 generates the socle in $A/(x_1, \ldots, x_d)A = A/m = K$. Hence, the image of $x_1^{t-1} \cdots x_d^{t-1}$ generates the socle in $A/(x_1^t, \ldots, x_d^t)A$ for all $t \geq 1$. By characterization (4) in the Theorem at the top of p. 3 of the Lecture Notes from October 24, it suffices to show that we cannot have an equation

$$(*) \quad x_1^{t-1} \cdots x_d^{t-1} = y_1 x_1^t + \cdots + y_d x_d^t$$

for any $t \geq 1$ with $y_1, \ldots, y_d \in R$.

Observe that if $A \to T$ is any ring homomorphism, $f_1, \ldots, f_h \in A$, and $I = (f_1, \ldots, f_h)A$, then $\mathrm{Tor}_1^A(A/I, T)$ is the quotient of the submodule of T^h whose elements are the relations on $f_1, \ldots, f_h$ with coefficients in T by the submodule generated by the images of the relations on $f_1, \ldots, f_h$ over A. (A free resolution of A/I over A begins

$$\cdots \longrightarrow A^k \xrightarrow{\alpha} A^h \xrightarrow{(f_i)} A \longrightarrow A/I \longrightarrow 0$$

where (f_i) is a $1 \times h$ row vector whose entries are $f_1, \ldots, f_h$ and the column space of the matrix α is the module of relations on $f_1, \ldots, f_h$ over A. Drop the A/I term, apply the functor $T \otimes \_$, and take homology at the T^h spot.)

Assume that we have the equation $(*)$ for some $t \geq 1$ and $y_1, \ldots, y_d \in R$ Let I be the ideal of A with generators

$$x_1^{t-1} \cdots x_d^{t-1}, \, x_1^t, \, \ldots, \, x_d^t.$$

Then

$$(\#) \quad (1, -y_1, \ldots, -y_d)$$

represents an element of $\mathrm{Tor}_1^A(A/I, R)$. Take S to be the residue class field L of R. $S = L$ is certainly regular. Hence, assuming the Vanishing Theorem for Maps of Tor, the image of the relation $(\#)$ in $\mathrm{Tor}_1^A(A/I, L)$ is 0. The image of the relation $(\#)$ has the entry 1 in its first coordinate. However, the relations on

$$x_1^{t-1} \cdots x_d^{t-1}, \, x_1^t, \, \ldots, \, x_d^t$$

over A all have first entry in m, since $x_1^{t-1} \cdots x_d^{t-1}, \notin (x_1^t, \ldots, x_d^t)A$. These relations map to elements of L^{d+1} whose first coordinate is 0, and so the image of $(\#)$ cannot be in their span, a contradiction. $\square$

It is worth noting that these two applications of the Vanishing Theorem for Maps of Tor are, in some sense, at diametrically opposed extremes. In the first application, the map to the regular ring S is a split injection. In the second, the map to the regular ring S is simply the quotient surjection of a local ring to its residue class field!