Math 711, Fall 2007 **Problem Set #1**

Due: Monday, October 8

1. Let R be a Noetherian domain of characteristic p. Let S be a solid R-algebra: this means that there is an R-linear map $\theta : S \to R$ such that $\theta(1) \neq 0$. Show that $IS \cap R \subseteq I^*$. Note that there is no finiteness condition on S.

2. Let S be weakly F-regular, and let $R \subseteq S$ be such that for every ideal of R, $IS \cap R = I$. Show that R is weakly F-regular.

3. Let M be a finitely generated module over a regular ring R of prime characteristic $p > 0$. Show that that for all $e \geq 1$, the set of associated primes of $\mathcal{F}^e(M)$ is equal to the set of associated primes of M.

4. Let (R, m, K) be a local ring of prime characteristic $p > 0$ with $\dim(R) = d > 0$. Let $I \subseteq J$ be two m-primary ideals of R such that $J \subseteq I^*$. Prove that there is a positive constant C such that $\ell(R/J^{[q]}) - \ell(R/I^{[q]})| \leq Cq^{d-1}$. (This implies that the Hilbert-Kunz multiplicities of I and J are the same.) [Here is one approach. Let h, k be the numbers of generators of I, J, respectively. Show that there exists $c \in R^\circ$ such that $cJ^{[q]} \subseteq I^{[q]}$ for $q \gg 0$, so that $J^{[q]}/I^{[q]}$ is a module with at most k generators over $R/(I^{[q]} + cR) = \overline{R}/\mathfrak{A}^{[q]}$ where $\overline{R} = R/cR$ and $\mathfrak{A} = I\overline{R}$. Moreover, $\dim(\overline{R}) = d - 1$ and $\mathfrak{A}^{qh} \subseteq \mathfrak{A}^{[q]}$.]

5. Let R be a reduced Noetherian ring of prime characteristic $p > 0$. Show that if $R/\mathfrak{p}_i$ has a test element for every minimal prime $\mathfrak{p}_i$ of R, then R has a test element.

6. Let (R, m, K) be a Cohen-Macaulay local ring, and let $x_1, \ldots, x_n$ be a system of parameters. Let $I_t = (x_1^t, \ldots, x_n^t)R$ for $t \geq n$ and let $I = I_1$.

(a) Prove that there is an isomorphism between the socle (annihilator of the maximal ideal) in R/I and the socle in R/I_t induced by multiplication by $x_1^{t-1} \cdots x_n^{t-1}$.

(b) Prove that an m-primary ideal J is tightly closed iff no element of $(J :_R m) - J$ is in J^*. Note that $(J :_R m)/J$ is the socle in R/J. (That R is Cohen-Macaulay is not needed here.)

(c) Prove that I is tightly closed in R if and only if I_t is tightly closed in R for every $t \geq 1$.