Math 711, Fall 2007 **Problem Set #1 Solutions**

1. Let $\theta(1) = c \neq 0$. If $r \in IS \cap R$, we have $r = f_1 s_1 + \cdots f_n s_n$ with $r \in R$, the $f_i \in I$, and $s_1, \ldots, s_n \in S$. Then for all q, $r^q = f_1^q s_1^q + \cdots f_n^q s_n^q$, and applying the R-linear map θ yields $cr^q = r^q\theta(1) = f_1^q\theta(s_1^q) + \cdots f_n^q\theta(s_n^q) \in I^{[q]}$. Hence, $r \in I^*$. $\square$

2. Since S is weakly F-regular, it is normal, and, hence, a finite product of weakly F-regular domains. It follows that R is reduced. We use induction first on the number of factors of R, if R is a product, and second on the number of factors of S. If R is not a domain, we can partition the minimal primes into two nonempty sets M_1 and M_2. We can construct a in all of the primes that are in M_1 and not in any of the primes that are in M_2, and b in all of the primes in M_2 and in none of the primes in M_1. Then $ab = 0$ and $a + b$ is not a zerodivisor in R. If we kill any minimal prime of S, either a or b becomes 0, and, in either case, a is in the ideal $(a + b)S$. Hence, a is in its tight closure and therefore in the ideal in S. Then $a \in (a + b)S \cap R = (a + b)R$, and so we can find $e \in R$ such that $a = e(a + b)$. Modulo every prime in M_1, we must have $e \equiv 0$, and modulo every prime in M_2 we must have $e \equiv 1$. It follows that $e \equiv e^2$ mod every minimal prime, and, hence, that e is a nontrivial idempotent in R. It is immediate that R is a product $Re \times Rf$ with $f = 1 - e$, and $Re \hookrightarrow Se$ and $Rf \hookrightarrow Sf$ inherit the hypothesis. Hence, by induction on the number of factors of R, both Re and Rf are weakly F-regular: consequently, so is R. Thus, we may reduce to the case where R is a domain. If R° maps into S°, which is automatic if S is a domain, then whenever $cr^q \in I^{[q]}$ for all $q \gg 0$ in R, we have that $cr^q \in I^{[q]}S = (IS)^{[q]}$ for all $q \gg 0$ in S, and then $r \in (IS)^*$ in S, i.e., $r \in IS$, since S is weakly F-regular. But then $r \in IS \cap R = I$, and so every ideal I of R is tightly closed.
Now suppose that $S = S_1 \times \cdots \times S_n$ where $n \geq 2$. We proceed by induction on n. Every S_i is an R-algebra. If $R \to S_i$ is injective for every i, then R° maps into $S^\circ = S_1^\circ \times \cdots \times S_n^\circ$. If not, we may assume by renumbering that $R \to S_n$ has a nonzero kernel P. Let $T = S_1 \times \cdots \times S_{n-1}$. We shall show that $R \to T$ still has the property that $IT \cap R = I$ for all $I \subseteq R$, and then the result follows by induction on n. Suppose not, and choose $I \subseteq R$ and $u \in R - I$ such that $u \in IT \cap R$. Let a be a nonzero element of P. Then $au \in aIT \cap R$, but $au \notin aI$ in R. Since au maps to 0 in S_n and $aIS = (0)$ in S_n, we have that $au \in aI(T \times S_n) = aIS$ as well, and so $au \in aIS \cap R - aI$, a contradiction. $\square$

3. If $P \in \mathrm{Ass}\,(M)$ then $R/P \hookrightarrow M$. Applying F^e, which is faithfully flat, we have $R/P^{[q]} \hookrightarrow F^e(M)$. Since $\mathrm{Rad}\,(P^{[q]}) = P$, P is a minimal prime of $P^{[q]}$, and so $R/P \hookrightarrow R/P^{[q]} \hookrightarrow F^e(M)$. Hence, $P \in \mathrm{Ass}\,\bigl(F^e(M)\bigr)$. Now suppose that $P \notin \mathrm{Ass}\,(M)$. Whether $P \in \mathrm{Ass}\,(M)$ or $P \in \mathrm{Ass}\,\bigl(F^e(M)\bigr)$ is unaffected by localization at P. (Note that if $T = R_P$, $F^e_T(M_P) \cong F^e_R(M)_P$.) Therefore, we may assume that (R, P, K) is local, and that $P \notin \mathrm{Ass}\,(M)$. Then there exists $x \in P$ that is not a zerodivisor on M. It follows that x^q is a nonzerodivisor on $F^e(M)$, and so $P \notin \mathrm{Ass}\,\bigl(F^e(M)\bigr)$, as required. $\square$

4. For every generator u_i of J we can choose $c_i \in R^\circ$ such that $cu_i^q \in I^{[q]}$ for all $q \gg 0$. Let $c \in R^\circ$ be the product of the c_i. Then $c \in R^\circ$ is such that $cJ^{[q]} \subseteq I^{[q]}$ for all $q \gg 0$. Now $0 \leq \ell(R/I^{[q]}) - \ell(R/J^{[q]})$ (since $I^{[q]} \subseteq J^{[q]}$), and this is the same as $\ell(J^{[q]}/I^{[q]})$. Since J has k generators, $J^{[q]}$ has at most k generators, and the same holds for $N_q = J^{[q]}/I^{[q]}$.

Since c and $I^{[q]}$ both kill N_q, we can map $R/(I^{[q]}+cR)^{\oplus k}$ onto N_q, which bounds its length by $k\ell\big(R/(I^{[q]}+cR)\big) = k\ell(\overline{R}/\mathfrak{A}^{[q]}) \le k\ell(\overline{R}/\mathfrak{A}^{qh})$. Since $\overline{R}$ has dimension $d-1$ and $\mathfrak{A}$ is primary to its maximal ideal, this length is bounded by $C_1(qh)^{d-1}$, using the ordinary Hilbert function. This yields the uppoer bound $kC_1h^{d-1}q^{d-1}$, so that we may take $C = kC_1h^{d-1}$. $\square$

5. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of R, and let $c_i \in R - \mathfrak{p}_i$ represent a test element in $R/\mathfrak{p}_i$. We may choose d_i not in $\mathfrak{p}_i$ but in all other minimal primes of R. Let $c = c_1d_1 + \cdots + c_nd_n$. Then $c \in R^\circ$, and if $u \in N^*_M$, this is true modulo every $\mathfrak{p}_i$, and so $c_iu \in N + \mathfrak{p}_iM$ for all i. Then $c_id_iu \in N$, since d_i kills $\mathfrak{p}_i$, and adding shows that $cu \in N$, as required. (The argument is valid both for test elements and for big test elements.) $\square$

6. This result is proved in the Lecture Notes of October 8.