Math 711, Fall 2007 **Problem Set #2**
Due: Wednesday, October 24

In all problems, R, S are Noetherian rings of prime characteristic $p > 0$.

1. Suppose that (R, m, K) has a test element. Show that for every proper ideal $I \subseteq R$,

$$I^* = \bigcap_n (I + m^n)^*.$$

2. Suppose that c is a completely stable test element in $(R,\, m,\, K)$. Let I be an m-primary ideal. Show that $u \in I^*$ if and only if $u \in (I\widehat{R})^*$ in $\widehat{R}$.

3. Let R be a Noetherian domain of characteristic $p > 0$, and suppose that the integral closure S of R in its fraction field is weakly F-regular. Prove that for every ideal I of R, $I^* = IS \cap R$. (This is the case in subrings of polynomial rings over a field K generated over K by finitely many monomials.)

4. If R is a ring of prime characteristic $p > 0$, define the *Frobenius closure* I^{F} of I to be the set of elements $r \in R$ such that for some $q = p^e$, $r^q \in I^{[q]}$. Suppose that $c \in R^\circ$ has the property that for every maximal ideal m of R, there exists an integer N_m such that c^{N_m} is a test element for R_m. Prove that for every ideal I of R, $cI^* \subseteq I^{\mathrm{F}}$.

5. Let $R \subseteq S$ be integral domains such that S is module-finite over R, and suppose that S has a test element. Prove that R has a test element.

6. Let $(R,\, m,\, K)$ be a local Gorenstein ring, and let $x_1,\, \dots\,, x_d$ be a system of parameters for R. Let $y \in R$ generate the socle modulo $(x_1,\, \dots\,, x_d)$. Suppose that for every integer $t \geq 1$, the ideal $\bigl(x_1^t,\, \dots\,, x_d^t,\, (x_1 \cdots x_d)^{t-1}y\bigr)R$ is tightly closed. Prove that either R is weakly F-regular, or else that $\tau(R) = m$.