Math 711, Fall 2007 **Problem Set #2 Solutions**

1. $\subseteq$ is clear. To prove $\supseteq$, if c is a test element and $u \in \bigcap_n (I + m^n)^*$ then for all q and n, $cu^q \in (I + m^n)^{[q]} = I^{[q]} + (m^n)^{[q]} \in I^q + m^n$. Fix q. Then $cu^q \in \bigcap_n (I^{[q]} + m^n) = I^q$. Hence, $u \in I^*$. $\square$

2. $u \in I^*$ iff $cu^q \in I^{[q]}$ for all q iff $cu^q \in I^q \widehat{R}$ for all q (since $\widehat{R}$ is faithfully flat over R, $J\widehat{R} \cap R = J$ for all $J \subseteq R$) iff $cu^q \in (I\widehat{R})^{[q]}$ for all q iff $u \in (I\widehat{R})^*$. It is not necessary that I be m-primary. More generally, if $R \subseteq S$, $c \in R$ is a test element for both rings, and $JS \cap R = J$ for all $J \subseteq R$, then $u \in I^*_R$ iff $u \in (IS)^*_S$. $\square$

3. Since R and S are domains, $R^\circ \subseteq S^\circ$ and $I^* \subseteq (IS)^* = IS$. Hence $I^* \subseteq IS \cap R$. The fact that $IS \cap R \subseteq I^*$ was proved in class (see the Theorem on the first page of the Lectcure Notes from October 12).

4. Frobenius closure of ideals commutes with localization: if W is a multiplicative system in R and $(u/w)^q \in (W^{-1}R)^{[q]}$, where $u \in R$ and $w \in W$, then for some $w_1 \in W$ we have $w_1 u^q \in I^{[q]}$, and then $(w_1 u)^q \in I^{[q]}$ as well. But then $w_1 u \in I^{\mathrm{F}}$, and so $u \in I^{\mathrm{F}}$, which shows that $u/w \in I^{\mathrm{F}} W^{-1}R$ as well. Now suppose that $u \in I^*$ but the $cu \notin I^{\mathrm{F}}$. We want to obtain a contradiction. The latter condition can be preserved by localizing at a maximal ideal m in the support of the image of cu in R/I^{F}. We then have that $u/1 \in (IR_m)^*$ in R_m, but that $u/1 \notin (IR_m)^{\mathrm{F}}$. Choose $q \geq N_m$. We also have that $u^q/1 \in \left((IR_m)^{[q]}\right)^*$, and so $c^q u^q/1 \in (IR_m)^{[q]}$, since c^q is a multiple of c^{N_m}. But this says that $(cu/1)^q \in (IR_m)^{[q]}$, which shows that $cu/1 \in (IR_m)^{\mathrm{F}}$, a contradiction. $\square$

5. Let c_S be a test element for S. Then c_S satisfies an equation of integral dependence on R whose constant term is not 0 (or factor out a power of x). Hence, c_S has a multiple c in R°. Now suppose that $u \in H^*_G$, where H is a submodule of the module G over R. Fix an R-linear map $\theta : S \to R$ whose value on 1 is nonzero: call the value d. θ induces an R-linear map $\eta : S \otimes G \to G$ such that $s \otimes g \mapsto \theta(s)g$: hence, if $g \in G$, $\eta(1 \otimes g) = dg$. We have $1 \otimes u \in \langle S \otimes_R H \rangle^*_{S \otimes_R G}$ and so $1 \otimes cu \in \langle S \otimes_R H \rangle$, i.e., $cu = \sum_{j=1}^h s_j \otimes h_j$, $s_j \in S$, $h_j \in H$. Apply η to obtain $cdu = \sum_{j=1}^n \theta(s_j) h_j \in H$. Thus, cd is a test element for R. $\square$

6. Let $x \in m$. It suffices to show that if $N \subseteq M$ are finitely generated modules and $u \in N^*_M$, then $xu \subseteq N$, for then $m \subseteq \tau(R)$ (if $\tau(R) = R$, R is weakly F-regular). If not, chose N' with $N \subseteq N' \subseteq M$ such that N' is maximal with respect to not containing xu. Then M/N' is a finite length essential extension of Rxu, which is killed by m. We may replace u and $N \subseteq M$ by $u + N'$ and $0 \subseteq M/N'$ as a counterexample. For large t, $I_t = (x_1^t, \ldots, x_d^t)R$ kills M, and since R/I_t is (R/I_t)-injective, M embeds in R/I_t. xu must map to a socle generator, which we may take to be the image, z, of $(x_1 \cdots x_d)^{t-1} y$. By hypothesis, $(I_t, z)R$ is tightly closed, so that Kz is tightly closed in R/I_t. But since $u \in 0^*_M$ its image $v \in 0^*_{R/I_t} \subseteq (Kz)^*_{R/I_t} = Kz$. Since $v \in Kz$, $xv = 0$. Since $M \subseteq R/I_t$, we also have $xu = 0$, a contadiction. $\square$