Math 711, Fall 2007 **Problem Set #3**
Due: Wednesday, November 7

1. Let R be a domain in which the test ideal has height two, and let P be a height one prime ideal of R. Suppose that $u \in N^*_M$, where $N \subseteq M$ are finitely generated R-modules. Let $R \to S$ be a homomorphism to a domain S with kernel P. Show that $1 \otimes u$ is in the tight closure of the image of $S \otimes_R N$ in $S \otimes_R M$ over S, i.e., in $\langle S \otimes_R N\rangle^*_{S\otimes_R M}$.

2. Let R be a Noetherian ring of prime characteristic $p > 0$.

(a) Let $I \subseteq R$ be an ideal, and let $W \subseteq R$ be a multiplicative system disjoint from every associated prime of every ideal of the form $I^{[q]}$. Show that $(IW^{-1}R)^*$ over $W^{-1}R$ may be identified with $W^{-1}I^*$.

(b) Let R be a reduced Cohen-Macaulay ring, and let $x_1, \dots, x_n$ be a regular sequence in R such that $I = (x_1, \dots, x_n)R$ is tightly closed in R. Let P be a minimal prime of I. Prove that R_P is F-rational. (Suggestion: first localize at the multiplicative system W consisting of the complement of the union of the minimal primes of I. After this localization, P expands to a maximal ideal.)

(c) Let (R, m, K) be F-rational and P a prime ideal of R. Show that R_P is F-rational.

3. (a) If M is a finitely generated module over a Noetherian ring R, show that M has a finite filtration such that every factor is a finitely generated torsion-free (R/P)-module N for some prime $P \in \mathrm{Ass}\,(M)$, and each such module N embeds in $(R/P)^{\oplus h}$ for some h.

(b) Let $R \to S$ be flat, where R and S are Noetherian rings, and let M be an R-module. Show that $\mathrm{Ass}_S(S \otimes_R M) = \bigcup_{P\in \mathrm{Ass}_R(M)} \mathrm{Ass}_S(S/PS)$.

4. Let (R, m, K) be a Gorenstein local ring of prime characteristic $p > 0$, and let $x_1, \dots, x_n$ be a system of parameters for R. Suppose that every x_i is a test element. Let $I = (x_1, \dots, x_n)R$. Show that $I :_R I^*$ is the test ideal $\tau(R)$ for R.

5. (a) Let (R, m, K) be a Gorenstein local ring of prime characteristic $p > 0$ that is F-finite or complete, and let $x_1, \dots, x_n$ be a system of parameters for R. Let $u \in R$ represent a generator of the socle in $R/(x_1, \dots, x_n)$. Show that R is F-split if and only if $u^p \notin (x_1^p, \dots, x_n^p)R$.

(b) Let K be a perfect field of characteristic $p > 0$, where $p \neq 3$. Determine for which primes p the ring $K[[x, y, z]]/(x^3 + y^3 + z^3)$ is F-split.

6. (a) Let R be ring of prime characteristic $p > 0$, and W be a multipllicative system in R, and let $S = W^{-1}R$. Let M be an S-module. Show that $\mathcal{F}^e_R(M) \cong \mathcal{F}^e_S(M)$.

(b) Let R be a Noetherian ring of prime characteristic $p > 0$. Show that if every submodule of every R-module is tightly closed, then the same holds for $W^{-1}R$ for every multiplicative system W in R. [Suggestion: it suffices to consider injective hulls of quotients of the ring by prime ideals.]