Math 711, Fall 2007 **Problem Set #3 Solutions**

1. Since the test ideal $\tau(R)$ has height two, it cannot be contained in the union of P and the minimal primes of R. Hence, we can choose a test element $c \in R^{\circ}$ that is not in P. If $u \in N^*M$, we have that $cu^q \in N_M^{[q]}$ for all $q \gg 0$. This is preserved when we apply $S \otimes_R \_$, and the image of c is not 0. $\square$

2. (a) Every element of I^* maps into $(IW^{-1}R)^*$, from which $I^*W^{-1}R \subseteq (IW^{-1}R)^*$. To show $\supseteq$, , let $r/w \in (IW^{-1}R)^*$. Then $r \in (IW^{-1}R)^*$ as well, so that for all $q \gg 0$, $(c/1)(r^q/1) \in (IW^{-1}R)^{[q]} = I^{[q]}W^{-1}R$. By the Proposition on p. 2 of the Lecture Notes from September 17, $W^{-1}R^{\circ}$ maps onto $(W^{-1}R)^{\circ}$, and so we may assume that $c \in R^{\circ}$. Then for all $q \gg 0$ there exists $w_q \in W$ such that $w_q c r^q \in I^{[q]}$. Since w is not in any associated prime of $I^{[q]}$, it is a nonzerodivisor on $I^{[q]}$. Then $cr^q \in I^{[q]}$, and $r \in I^*$. $\square$

(b) For every q, $x_1^q, \dots, x_n^q$ is a regular sequence in the Cohen-Macaulay ring R, and so $R/I^{[q]}$ is Cohen-Macaulay, which implies that all associated primes of $I^{[q]}$ are minimal primes of $I^{[q]}$ and, hence, of I. By part (a), we can localize at W, the complement of the union of the minimal primes of I, and $IW^{-1}R$ will be tightly closed, while $PW^{-1}R$ is a maximal idea of $W^{-1}R$. Thus, I remains tightly closed when we localize at $PW^{-1}R$, by the Lemma on p. 2 of the Lecture Notes from September 17. We have a reduced Cohen-Macaulay local ring in which a system of parameters generates a tightly closed ideal, and the ring is F-rational by the Theorem on p. 9 of the Lecture Notes from October 8. $\square$

(c) R is F-rational and so it is certainly Cohen-Macaulay and reduced. Choose part of a system of parameters for R that is also a system of parameters for R_P. The ideal these elements generate in R is tightly closed, and, hence, so is the ideal they generate in R_P. $\square$

3. Let P be maximal in $\mathrm{Ass}\,(M)$ and let $N = \mathrm{Ann}_M P$. Then $N \neq 0$, and it will suffice to show that $(*)$ $\mathrm{Ass}\,(M) = \mathrm{Ass}\,(N) \cup \mathrm{Ass}\,(M/N)$. In fact, $\mathrm{Ass}\,(N) \subseteq \mathrm{Ass}\,(M)$ and any prime in $\mathrm{Ass}\,(N)$ must contain P. It follows that $\mathrm{Ass}\,(N) = \{P\}$, since P is maximal in $\mathrm{Ass}\,(M)$, and that N is a torsion-free module over R/P. If $(*)$ holds, then by Noetherian induction on M, M/N will have a filtration as required whose inverse image in M, together with N, will give the required filtration of M. To prove $(*)$ note that we always have $\subseteq$ and that $\mathrm{Ass}\,(N) \subseteq \mathrm{Ass}\,(M)$. We need only show that $\mathrm{Ass}\,(M/N) \subseteq \mathrm{Ass}\,(M)$. Suppose $Q \in \mathrm{Ass}\,(M/N)$. If P is not contained in Q, then $N_Q = 0$, and so $M_Q = (M/N)_Q$. Thus, $QR_Q \in \mathrm{Ass}\,(M_Q)$ over R_Q, and so $Q \in \mathrm{Ass}\,(M)$, as required. Now suppose that $P \subset Q$ strictly, and that Q is the annihilator of $u \in M - N$. Since $u \notin N$, we can choose $r \in P$ such that $ru \neq 0$. But then $Qru = rQu \subseteq rN = 0$, and so Q or a larger prime is in $\mathrm{Ass}\,(M)$, contradicting the maximality of P. $\square$

(b) If $P \in \mathrm{Ass}_R(M)$, then $R/P \hookrightarrow M$, and so $S/PS = S \otimes_R R/P$ injects into $S \otimes_R M$. Thus, $\mathrm{Ass}_S(S/PS) \subseteq \mathrm{Ass}\,_S(S \otimes_R M)$ for every $P \in \mathrm{Ass}_R(M)$. This proves $\supseteq$. For the other direction, suppose that we have a finite filtration of M such that every factor N_i is a torsion-free module over R/P_i for some $P_i \in \mathrm{Ass}_R(M)$. Then we have a filtration of $S \otimes_R M$ by the modules $S \otimes N_i$. It suffices to show that if N_i is nonzero torsion-free over R/P_i, then $\mathrm{Ass}_S(S \otimes_R N_i) \subseteq \mathrm{Ass}_S(S/P_iS)$. But if N_i is torsion-free over R/P_i, it embeds in a free module $(R/P_i)^{\oplus h}$ by the Lemma on the first page of the Lecture Notes from October 12. Therefore, $S \otimes_R N_i$ embeds in $S \otimes_R (R/P)^{\oplus h}$, which is a direct sum of copies of S/PS. Hence, $\mathrm{Ass}\,_S(S \otimes_R N_i) \subseteq \mathrm{Ass}_S(S/P_iS)$. $\square$

4. Evidently $\tau(R) \subseteq I :_R I^*$ for every ideal I. In the local case, if I_t is a sequence of m-primary irreducible ideals cofinal with the powers of m, then $\tau(R) = \bigcap_t (I_t :_R I_t^*)$. To see this, suppose c is in the intersection but not in $\tau(R)$. Then there exist finitely generated modules $N \subseteq M$ such that $u \in N^*$ but $cu \notin N$. Replace N by N' maximal such that $N \subseteq N' \subseteq M$ and $cu \notin N'$. Then replace, u, M, N' by v, $M/N', 0$, where v is the image of u in M/N'. Every nonzero submodule of M contains $Rcv = Kcv$, and so M is a finite length essential extension of K and is killed by some I_t: then $M \hookrightarrow R/I_t$ with cv corresponding to the socle element. Then v corresponds to an element r of I_t^*/I_t with $cr \notin I_t$, a contradiction, since $cI_t^* \subseteq I_t$ by hypothesis. If $x_1, \ldots, x_n$ is generated by test elements in the Gorenstein ring (R, m, K), let $I_t = (x_1^t, \ldots, x_n^t)$. We only need that $I :_R I^* = I_t :_R I_t^*$ for all t. Let $u \in I_t^*$. Then $x_i u \in I_t$ for all i, and $u \in I_t :_R I = I_t + y^{t-1}R$, where $y = x_1 \cdots x_n$. But if $u = f + y^{t-1}r$, then $r \in I_t^* :_R y^{t-1} = J$: we'll show that $J \subseteq I^*$. Since $y^{t-1}r \in I_t^*$, for some $d \in R^\circ$ and all $q \gg 0$, $dy^{q(t-1)}r^q \in (I_t)^{[q]} = I_q \Rightarrow dr^q \in I_{tq} :_R y^{tq-q} = I^{[q]}$. Thus, $r \in I^*$, and $I_t^* = I_t + y^{t-1}I^*$. But then $cI_t^* \subseteq I_t$ iff $cy^{t-1}I^* \subseteq I_t$, and this holds iff $cI^* \subseteq I_t :_R y^{t-1} = I$. Thus, $I_t^* :_R I_t = I^* :_R I$, as required. $\square$

5. (a) With R F-finite or complete, it follows from part (6) of the splitting crriterion at the top of p. 3 of the Lecture Notes from October 24 that the map is split iff $uS \notin (x_1, \ldots, x_n)S$, where S is R viewed as an R-algebra via $F : R \to R$. But this means precisely that $u^p \notin (x_1^p, \ldots, x_n^p)R$.

(b) Use x, y as a system of parameters. The socle mod (x, y) is represented by z^2. Note that R is free over $A = K[[x, y]]$ on the basis 1, z, z^2, and so every element can be uniquely represented as $a_0 + a_1 z + a_2 z^2$ with the $a_i \in A$. The ring is F-split if and only if $z^{2p} \notin (x^p, y^p)$. Let $p = 3k + r$, where $r = 1$ or 2 and so $2p = 6k + 2r$. Then $z^{2p} = z^{6k} z^{2r} = \pm(x^3 + y^3)^{2k} z^2$ if $r = 1$ and $\pm(x^3 + y^3)^{2k+1} z$ if $r = 2$. This is in (x^p, y^p) iff all terns of $(x^3 + y^3)^{2k}$ (resp., $(x^3 + y^3)^{2k+1}$) are: the binomial coefficients do not vanish, since $2k + 1 < p$. In case $r = 1$ there is a term involving $x^{3k} y^{3k}$ which is not in (x^p, y^p). In the second case, every term involves an exponent on x^3 or on y^3 that is at least $k + 1$, and $3k + 3 > p$. Hence, the ring is F-split if $p \equiv 1 \bmod 3$ and is not F-split if $p \equiv 2 \bmod 3$.

6. (a) When N is already a module over $W^{-1}R = S$, $S \otimes_R N \cong N$ as S-modules. Hence, $\mathcal{F}_S^e(M) \cong \mathcal{F}_S^e(S \otimes_R M) \cong S \otimes_{\mathbb{R}} \mathcal{F}_R^e(M)$ (by property (11) on p. 3 of the Lecture Notes from September 12), and this is simply $\mathcal{F}_R^e(M)$ because every $w \in W$ already acts invertibly on this module, since w^q does.

(b) By the equivalent condition (c) in the Proposition on p. 3 of the Lecture Notes from October 22, it suffices to show that for every prime ideal Q of $W^{-1}R$, 0 is tightly closed in the injective hull of $W^{-1}R/Q$. We know that Q has the form $PW^{-1}R$ for some prime ideal P of R disjoint from W. Thus it suffices to check that 0 is tightly closed in $E = E(R/P)$ thought of as an $W^{-1}R$ module. Suppose to the contrary that $u \neq 0$ is in the tight closure of 0. By the Proposition on p. 2 of the Lecture Notes from September 17, we may assume the element used in establishing that u is in the tight closure is the image of $c \in R^\circ$. Then cu^q is 0 in $\mathcal{F}_S^e(E)$ for all $q \gg 0$. Since $\mathcal{F}_S^e(E) \cong \mathcal{F}_R^e(E)$, this shows that u is in the tight closure of 0 in E working over R, a contradiction. $\square$