Math 711, Fall 2007 **Problem Set #4 Solutions**

1. Let ${}^eR$ indicate R and viewed as an R-algebra via F_R^e and similarly for ${}^eS$. We have that $\mathcal{F}_R^e(M) \otimes_R \mathcal{F}_S^e(N) = ({}^eR \otimes_R M) \otimes_{{}^eR} ({}^eS \otimes_S N) \cong (M \otimes_R {}^eR) \otimes_{{}^eR} ({}^eS \otimes_S N)$ (by the associativity of $\otimes$) $\cong M \otimes_R ({}^eS \otimes_S N) \cong ({}^eS \otimes_S N) \otimes_R M \cong {}^eS \otimes_S (N \otimes_R M)$ (by the associativity of $\otimes$) $\cong {}^eS \otimes_S (M \otimes_R N) = \mathcal{F}_S^e(M \otimes_R N)$. Then, as we trace through the identifications, $u^q \otimes v^q \mapsto (1 \otimes u) \otimes (1 \otimes v) \mapsto (u \otimes 1) \otimes (1 \otimes v) \mapsto u \otimes (1 \otimes v) \mapsto u \otimes (v \otimes 1) \mapsto (u \otimes v) \otimes 1 \mapsto 1 \otimes (u \otimes v)$, as required. $\square$

2. If x is a nonzerodivisor on R, it is also on the smallest ideal $\neq 0$ in the specified filtration (or on R/I, if $I = 0$), which is $\cong (R/I)^t$, $t > 0$. Thus, x is a nonzerodivisor on R/I. If x is a nonzerodivisor on R/I, it is also on each I_{t-1}/I_t, since these are (R/I)-free. By the Proposition on p. 1, October 8, and its proof, x is a nonzerodivisor on R, and the filtration is preserved when we kill x. This proves (a). Since I is nilpotent, $\dim(R/I) = \dim(R)$ $\Rightarrow$ the image of a system of parameters for R is a system of parameters for R/I. The statement about the Cohen-Macaulay property in (b) now follows at once from part (a), and the fact that as we successively kill elements of a system of parameters, we are in the same situation that we had initially, but with the dimension one smaller. $\square$

3. (a) Localize at an element in all minimal primes except P so that P is the only minimal prime. Any finitely generated (R/P)-module has a localization at one element not in P that is free. We may apply this successively to each factor in a filtration as discussed in the statement of the problem. Now, when we localize at any prime containing P, we are in the situation of Problem 2, and part (b) gives the required result. $\square$

(b) We apply conditions (1) and (2°) of the criterion at the top of p. 3 of the Lecture Notes from November 5. We know that if $Q \subseteq P$ and R_P is Cohen-Macaulay then R_Q is Cohen-Macaulay. Now suppose that R_P is Cohen-Macaulay and we want an open neighborhood U of P in $\mathcal{V}(P)$ such that R_Q is Cohen-Macaulay for $Q \in U$. Choose a maximal regular sequence in PR_P consisting of images of elements $x_1, \ldots, x_h$ in R. After localizing at one element of R_P, these form a regular sequence in R, and we may kill them without affecting the issue. Thus, we may assume that P is a minimal prime of R. After localizing at one more element of $R - P$, we have from part (a) that P is the only minimal prime of R and that R_Q is Cohen-Macaulay iff $(R/P)_Q$ is Cohen-Macaulay. Thus, we need only consider the case where $R = R/P$ is a domain. Suppose that $R = S/\mathfrak{p}$ where $\mathfrak{p}$ is is a prime ideal of the Cohen-Macaulay ring S. After localizing at one element of $S - \mathfrak{p}$, $\mathfrak{p}$ will contain a maximal regular sequence that is also a maximal regular sequence in $S_\mathfrak{p}$. We may kill it without affecting the issue. We now have that $\mathfrak{p}$ is a minimal prime of the Cohen-Macaulay ring S. By another application of part (a), we may localize at a single element of $S - \mathfrak{p}$ so that $\mathfrak{p}$ is the only minimal prime of S and S_Q is Cohen-Macaulay iff $(S/\mathfrak{p})_Q$ is Cohen-Macaulay. But then $S/\mathfrak{p}$ is Cohen-Macaulay. $\square$

4. Since S is a torsion-free finitely generated R-module it is embeddable in a finitely generated free R-module R^h. The image of 1 in R^h will have at least one nonzero coordinate, and the composite of $S \hookrightarrow R^h$ and projection on that cooridinate will give an R-linear map $\theta : S \to R$ such that $\theta(1) = c \in R^\circ$. Now suppose that $u \in \langle S \otimes_R N \rangle$ in $S \otimes_R M$.

Then there exists $d \in S^\circ$ such that $du^q \in \langle S \otimes_R N\rangle^{[q]} = \langle S \otimes_R N^{[q]}\rangle$ for all $q \gg 0$. We may replace d by a nonzero multiple R°, and so we may assume that that $d \in R^\circ$. Now apply $\theta \otimes \mathbf{1}_{\mathcal{F}^e(M)}$ to obtain that $c(du^q) \in N^{[q]}$ for all $q \gg 0$, as required. $\square$

5. Write R as a finite-module over a DVR A: it is A-free. Let $M^\natural = \mathrm{Hom}_A(M,\, A)$. The rank of M over R is its rank over A divided by the rank of R over A, and the same holds for $M^\natural$. Then $M^\natural$ and M have the same rank over R. It suffices to show that the type of M is the number of generators of $M^\natural$ over R. Let x be a regular parameter for A. Let $N = M/xM$. Then $M^\natural/xM^\natural \cong \mathrm{Hom}_K(N,\, A/xA) = \mathrm{Hom}_K(N,\, K)$. Let $u_1,\, \dots\, , u_h$ generate the maximal ideal of R/xR. Let $\_^*$ indicate duals into K. We have an exact sequence $0 \to V \to N \to N^h$ where $\dim_K(V)$ is the type of M/xM and of M and the map has matrix $(u_1 \ \dots \ u_h)$. Applying $\mathrm{Hom}_K(\_\,, K)$ gives an exact sequence which shows that $V^* \cong N^*/(u_1,\, \dots\, , u_h)N^* \cong M^\natural/mM^\natural$. $\square$

6. Again apply (1) and (2°): we know (1). Since the Cohen-Macaulay locus is open, we may assume that the ring is Cohen-Macaulay. Suppose the type of R_P is t. We may also localize at $c \in R - P$ so that R/P is Cohen-Macaulay. As in 3., we may assume P is the only minimal prime of R. If $R = S/J$, with S Gorenstein, then $R/P = S/Q$ for a prime ideal Q of S. After localizing $c \in S - Q$, we may kill a maximal regular sequence for S_Q that is is in J. After localizaing at $c' \in S - Q$ we may assume that S is Gorenstein with unique minimal prime Q, and that $R = S/J$ with $J \subseteq Q$. Since R_P has type t, we have an S_Q-module embedding $\iota : R/P \hookrightarrow (S/Q)^{\oplus t}$: embed the socle of R_P and then extend the map to R_P using that S_Q is injective. Since localization commutes with Hom, we have $R \hookrightarrow S^t$ inducing $\alpha\iota$, where α is a unit of S_Q. Then $0 \to R \to S^t \to C \to 0$ is exact where each S-module is killed by a power of Q. Filter R and C so that the finitely many factors are (S/Q)-modules, and localize at $c'' \in S - Q$ so that all factors are free over $S/Q = R/P$. Then all of R, C, S^t are maximal Cohen-Macaulay modules. If we localize at any prime $\mathcal{M}$ of S and kill a system of parameters $y_1,\, \dots\, , y_k$, we get an exact sequence $0 \to R_{\mathcal{M}}/(y_1,\, \dots\, , y_k)R_{\mathcal{M}} \to \big(S_{\mathcal{M}}/(y_1,\, \dots\, , y_k)S_{\mathcal{M}}\big)^{\oplus t} \to C_{\mathcal{M}}/(y_1,\, \dots\, , y_k)C_{\mathcal{M}} \to 0$, and since the socle in the middle term has dimension at most t, so does the socle in $R/(y_1,\, \dots\, , y_k)R$. If R is excellent, localize at $c \in R-P$ so that R/P is regular. Let $M_i = \mathrm{Ann}_R P^i$ and $M = M_1$. Localize at $c' \in R - P$ so that all the M_{i+1}/M_i are (R/P)-free. Then $M \cong (R/P)^{\oplus t}$, since M is R/P-free, and its rank is t because R_P has type t. For each prime Q of R, let $B = {}^QB$ be the quotient of R_Q by the images of elements $x_1,\, \dots\, , x_K \in R$ that form a regular system of parameters in $(R/P)_Q$. We shall write $\__B$ for $B \otimes_R \_$. $PR_B = PB$ is the maximal ideal of $R_B = B$, and is nilpotent. B/PB is a field. We show by induction on n that, after localizing at $c'' \in R - P$, then $(*)$ for $Q \in \mathrm{Spec}\,(R)$, $\mathrm{Ann}_B P^i$ is $(M_i)_B$. In particular, the socle of B is $(M_1)_B \cong (B/PB)^t$, and so the type is t, as required. By the induction hypothesis, we may assume that we have localized so that $(*)$ holds for R/M_1. We know $M_1 = (R/P)^{\oplus t}$. We have an exact sequence $(*)\quad 0 \to M_1 \to R \to R/M_1 \to 0$ which yields $(\#)\quad 0 \to \mathrm{Hom}_R(R/P,\, M_1) \xrightarrow{f} \mathrm{Hom}_R(R/P,\, R) \to \mathrm{Hom}_R(R/P,\, R/M_1) \xrightarrow{\partial} \mathrm{Ext}^1_R(R/P,\, M_1) \to D \to 0$, where $D = \mathrm{Coker}\,(\partial)$. The map f is onto and so ∂ is injective. We want to compare the result of applying $\__B$ to $(\#)$ with the result of forming the long exact sequence $(\#_B)$, , truncated at the same spot, for $\mathrm{Hom}_B\big(B/PB,\, \_\big)$ applied to the short exact sequence obtained by applying $\__B$ to $(*)$. We want these two to be

the same. It suffices to localize at $c''' \in R - P$ so the map ∂_B in $(\#_B)$ is still injective, and $\_\,_B$ commutes with the formation of $\mathrm{Hom}_R(R/P,\, R/M_1)$ and $\mathrm{Ext}^1(R/P,\, M)$ as well as the formation of the connecting homomorphism. For then the counterpart f_B of f in $(\#_B)$ will again be onto. This can be achieved by localizing to make sufficiently many (but finitely many) auxiliary modules have finite filtrations by free (R/P)-modules, along with the observation that for $\mathrm{Hom}_R(R/P,\, R/M_1)$, we get what we need from the induction hypothesis. Localization at Q and killing a regular system of parameters in R_Q then preserve exactness. Let $G_2 \to G_1 \to G_0 \to R/P \to 0$ be part of a finite free R-resolution for R/P. Then we also have exact sequences $0 \to Z_1 \to \mathrm{Hom}_R(G_1,\, M_1) \to \mathrm{Hom}_R(G_2,\, M_1) \to E \to 0$, $0 \to Z_0 \to \mathrm{Hom}_R(G_0,\, M_1) \to B_1 \to 0$, and $0 \to B_1 \to Z_1 \to \mathrm{Ext}^1_R(R/P,\, M_1) \to 0$, If we localize so that every module occurring has a finite filtration by free R/P-modules, then $x_1, \ldots, x_k$ is a regular sequence on every module after localization at Q, and the exactness is preserved when we apply $\_\,_B$. Moreover, $\_\,_B$ commutes with Hom from a free R-module. The fact needed about the connecting homomorphism is now easy to check. $\square$