Math 711, Fall 2007 **Problem Set #5**
Due: Wednesday, December 12

1. Let K be a field of characteristic $p > 0$ with $p \neq 3$. Let X, Y, and Z be indeterminates over K, and let $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3) = K[x, y, z]$.

(a) Show that R is module-finite, torsion-free, and generically étale over $A = K[x, y]$, and find the discriminant D of R over A with respect to the basis $1, z, z^2$.

(b) Use the Jacobian ideal to show that $(x^2, y^2, z^2) \subseteq \tau_{\mathrm{b}}(R)$.

2. Continue the notation of problem 1. Show that xyz^2 is a socle generator modulo $I = (x^2, y^2)R$. Show that $I^* = (x^2, y^2, xyz^2)R$. Conclude using problem 4. of Problem Set #3 that the test ideal of R is m.

3. Let $x_1, \ldots, x_k$ be part of a system of parameters in an excellent local domain (R, m, K) of prime characteristic $p > 0$, and let $I = (x_1, \ldots, x_k)R$. Prove that for every multiplicative system W in R, $(IW^{-1}R)^* = I^*W^{-1}R$. That is, tight closure commutes with localization for such an ideal I.

4. Let R be a locally excellent domain of prime characteristic $p > 0$ that is a direct summand of every module-finite extension domain. Prove that R is F-rational. In particular, it follows that R is Cohen-Macaulay.

5. Let (R, m, K) be a complete weakly F-regular local ring of Krull dimension d of prime characteristic $p > 0$. Let S be a Noetherian R-algebra such that the height of mS in S is d. Prove that R is a direct summand of S.

6. Let R be an $\mathbb{N}$-graded domain over an algebraically closed field K of prime characteristic $p > 0$, where $R_0 = K$, and let I be an ideal generated by forms of degree $\geq d \geq 1$. Let G be a form of degree d that is not in I. Prove that $G \notin I^*$.