Math 711, Fall 2007 **Problem Set #5 Solutions**

1. (a) The extension is module-finite since the equation is monic in Z, and generically étale since adjoining a cube root gives a separable field extension in characteristic $\neq 3$. The matrix has as entries the traces of the elements $z^{i-1+j-1}$, $1 \leq i,\, j \leq 3$. The trace of 1 is 3, and the trace of multiplication by z or z^2 is 0 (the matrices have all zeros on the diagonal). Since $z^3 = -(x^3+y^3)$ and $z^4 = -(x^3+y^3)z$, the matrix of traces is $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & -3(x^3+y^3) \\ 0 & -3(x^3+y^3) & 0 \end{pmatrix}$, whose determinant is $-27(x^3+y^3)^2 = -27z^6$.

(b) The partial derivatives are $3X^2$, $3Y^2$, and $3Z^2$, and 3 is invertible in K. $\square$

2. R is Gorenstein, and not weakly F-regular, since $z^2 \in (x,\, y)^*$. (The solution for 5(b) in Problem Set #3 gives $x(z^2)^q \in (x^q,\, y^q)$ for all q.) Hence, no parameter ideal is tightly closed in R. Since z^2 is a socle generator mod (x, y), $(xy)z^2$ is mod $(x^2,\, y^2)$. Mod $J = (x^2,\, y^2,\, xyz^2)$ the ring is K-spanned by $1,\, x,\, y,\, z,\, xy,\, xz,\, yz,\, z^2,\, xz^2,\, yz^2$ and xyz. It suffices to show that $J = I^*$: since $I \neq I^*$, $xyz^2 \in I^*$. If not, there exists $v \in I^*$ representing a socle element of $R/J - \{0\}$. The socle in R/J is spanned by the images of xz^2, yz^2, and xyz. Hence, it suffices if when $a,\, b,\, c \in K$, not all 0, then for some q, $x^2(axz^2 + byz^2 + cxyz)^q \notin (x^{2q},\, y^{2q})$. The left hand side is $A + B + C$ where $A = a^q x^{q+2}(x^3+y^3)^k z^\rho$, $B = b^q x^2 y^q (x^3+y^3)^k z^\rho$, and $C = c^q x^{q+2} y^q (x^3+y^3)^h z^{\rho'}$, where $2q = 3k + \rho$, $q = 3h + \rho'$, with $1 \leq \rho,\, \rho' \leq 2$. R is $K[x, y]$-free on the basis $1, z, z^2$. Since $\rho \equiv 3\rho' \mod 3$, terms from C cannot cancel those from A or B. Exponents on x in terms from A are $\equiv q+2 \mod 3$; those from B are $\equiv 2 \mod 3$: these cannot cancel either.

Thus, if $az^2x + bz^2y + cxyz \in I^*$, each term with nonzero coefficient $\in I^*$. Thus, it suffices to show that each of $z^2x, z^2y,\, zxy \notin I^*$. Say $zxy \in I^*$. Then colon-capturing (cf. the Theorem, bottom of p. 2, November 12) gives $z \in (x^2, y^2)^* : xy \subseteq \big((x^2, y^2) : xy\big)^* = (x, y)^*$. But $x^2z^q = x^2(x^3+y^3)^h z a^\rho$, and if $q = p$, $x^2(x^3+y^3)^h$ has a term $x^2 x^{3\lfloor h/2 \rfloor} y^{3(h - \lfloor h/2 \rfloor)}$. Since $3\lfloor h/2 \rfloor \leq 3h/2 \leq p/2 < p$ and $3(h - \lfloor h/2 \rfloor) \leq 3\big(h - (h-1/2)\big) \leq 3h + 3/2 < p+1 < 2p$, and $\binom{k}{\lfloor h/2 \rfloor} \neq 0$ with $k < p$, $z \notin (x^2, y^2)$.

If $xz^2 \in (x^2,\, y^2)^*$, then $z^2 \in (x^2,\, y^2)^* : x \subseteq \big((x^2, y^2) : x\big)^* = (x^2,\, y)^*$. We will show this is false. (By symmetry, this handles $v = yz^2$.) Then $x^2 z^q \in (x^{2q}, y^q)$ for all q $\Rightarrow x^2(x^3+y^3)^k z^{\rho'} \in (x^{2q}, y^q)$ and so $x^2(x^3+y^3)^k \in (x^{2q}, y^q)$. But $\binom{k}{1} x^2 (x^3)^{k-1} y^3 \notin (x^{2q},\, y^q)$: since p does not divide $k = (2q - \rho')/3$ the coefficient is nonzero, while the degree in x is $2 + 3k - 3 = 3k - 1 < 2q$ and the degree in y is $3 < q$ for all $q \gg 0$.

It follows that the test ideal in R_m where $m = (x, y, z)R$ is $(x^2,\, y^2)R_m : JR_m = mR_m$, i.e., the annihilator of 0^*_E in the injective hull of R_m/mR_m is m. This is also true for $E_R(R/m) \cong E$. The localization at other maximal ideals is regular, and the annihilator of $0^*_{E'} = 0$ in the injective hull E' of R/m' for any other maximal ideal m' of R. It follows from the Theorem at the top of p. 5, notes from Novemberr 30, that $\tau(R) = m$. $\square$

3. Suppose $u \in R$ (we may assume this after clearing denominators) has the property that its image in $W^{-1}R$ is in $(IW^{-1}R)^* - (I^*)W^{-1}R$. Choose a prime ideal of $W^{-1}R$ that

contains $(I^*)W^{-1}R :_{W^{-1}R} u \supseteq IW^{-1}R$. Localizing at this prime gives a counterexample in which $W = R - P$ for some prime P with $I \subseteq P$. But then I is generated by part of a system of parameters for R_P, and we may replace tight closure by plus closure throughout. Since plus closure commutes with localization, the result follows.

4. We need to prove that every ideal I generated by parameters in R is tightly closed. But if $u \in R$ and $u \in I^* = I^+$, there is a module-finite extension S of R such that $u \in IS \cap R$. Since $R \hookrightarrow S$ splits by hypothesis, $IS \cap R = I$. $\square$

5. Since R is complete, it suffices to prove that ideals I of R are contracted from S. Suppose $u \in IS \cap R$. Since R is weakly F-regular, it suffices to show that $u \in I^*$. Hence, it suffices to show that S is a solid R-algebra, i.e., that $H^d_m(S) \neq 0$, or $H^d_{mS}(S) \neq 0$. Since height $(mS) = d$, we can pick a minimal prime $\mathcal{M}$ of mS of height d. Then $\mathcal{M}S_{\mathcal{M}} = \mathrm{Rad}\,(mS_{\mathcal{M}})$, and so $H^d_{mS}(S)_{\mathcal{M}} = H^d_{mS}(S_{\mathcal{M}}) = H^d_{\mathcal{M}S_{\mathcal{M}}}(S_{\mathcal{M}}) \neq 0$, since $\dim(S_{\mathcal{M}}) = d$. $\square$

6. Assume $(*_q)\quad cG^q = u_{q1}G_1^q + \cdots u_{qh}G_h^q$ for $q \gg q_1$ where $G, G_1, \ldots, G_h$ all have the same degree and $c \neq 0$. The same holds when we pass to a homogenous component $c_k \neq 0$ of c and the degree k components of the u_i. Thus, we may assume that c and all $u_{qh} \in [R]_k$. Take h minimum. If all of the $u_{qi} \in Kc$, divide by c and take q th roots to show $G \in I$. Hence, for every q, at least one $u_{qi} \notin cK$. Choose i such that $u_{qi} \notin Kc$ for $q \in \mathcal{Q}$ with $|\mathcal{Q}| = \infty$. By renumbering, say $i = h$, so that $u_{qh} \notin Kc$ for $q \in \mathcal{Q}$. It suffices if there exist q_0 and $c' \in R^\circ$, such that for $q \subseteq \mathcal{Q}$, there exists an R^{q_0}-lineaar map $\theta_q : R \to R$ such that $\theta_q(c) = c'$ and $\theta_q(u_{hq}) = 0$. For then applying θ_q for $q \geq \max\{q_0, q_1\}$ with $q \in \mathcal{Q}$ to $(*_q)$ shows that $c'G^q \in (G_1, \ldots, G_{h-1})^{[q]}$ for infinitely many q, and we may replace h by $h-1$. Let $\mathcal{K} = \mathrm{frac}\,(R)$ and $K' = \bigcap_q \mathcal{K}^q$. Then $K' = K$: to see this, choose a separating transcendence basis for $\mathcal{K}$ and enlarge $\mathcal{K}$ to be Galois over a pure transcendental extension $\mathcal{F} = K(y_1, \ldots, y_d)$. Given $w \in K' - K$, all elementary symmetric functions of its conjugates over $\mathcal{F}$ are also in K', and so one of them $z \in \mathcal{F} \cap K' - K$. Write $z = (f/g)^q$ with $f, g \in K[\underline{y}]$ in lowest terms and not all exponents divisible by p. Then z has a pq th root in $\mathcal{K}$, and so f/g has a p th root in $\mathcal{K}$ not in $\mathcal{F}$, contradicting that $\mathcal{K}/\mathcal{F}$ is Galois.

Let $v_1, \ldots, v_N$ be a K-basis for $[R]_K$. Then $\dim_{\mathcal{K}^q}\mathrm{span}_{\mathcal{K}^q}\{v_1, \ldots, v_n\}$ cannot decrease with q. Pick q_0 for which it is maximum, and renumber so that $v_1, \ldots, v_t$ give a basis for the span over $\mathcal{K}^{q_0}$. If $t < N$, write v_{t+1} as a $\mathcal{K}^{q_0}$-linear combination of $v_1, \ldots, v_t$. Then we must have that some coefficient is not in K, and we can choose $q' > q_0$ so that this coefficient is not in $\mathcal{K}^{q'}$. Then $v_1, \ldots, v_{t+1}$ are independent over $\mathcal{K}^{q'}$: a new relation would give a relation on $v_1, \ldots, v_t$. Thus, $v_1, \ldots, v_N$ are independent over $\mathcal{K}^{q_0}$. Extend $v_1, \ldots, v_N$ to $v_1, \ldots, v_B \in R$, a $\mathcal{K}^{q_0}$-basis for $\mathcal{K}$ over $\mathcal{K}^{q_0}$. Then we can choose $d \in R^{q_0} - \{0\}$ such that $dR \subseteq M = \sum_{j=1}^B R^{q_0}v_j$. We can now define θ_q as follows. Choose a K-linear map $T : V \to R$ that sends c to c, and kills u_h: this is possible since c and u_h are linearly independent over K. Extend this map to $M = \sum_{j=1}^B R^{q_0}v_j \to R$: let the values on $v_1, \ldots, v_N$ be given by T, and choose the values on the other v_j arbitrarily. This gives $\eta_q : M \to R$ that is R^{q_0}-linear such that $\theta_q(c) = c$ and $\eta_q(u_{qh}) = 0$. Finally, define θ_q on R by $\theta_q(r) = \eta_q(dr)$. Then $\theta_q(c) = \eta_q(dc) = d\eta_q(c) = dc$, and $\theta_a(u_{hq}) = \eta_q(du_{hq}) = d\eta_q u_{hq}) = 0$. Take $c' = dc$. $\square$