Commutative Algebra Seminar: Lecture of January 9, 2003

The following is a summary of where we were.

Let R be an excellent bi-equidimensional semi-local ring of prime characteristic $p > 0$. We shall say that such a ring R is *standard*. Let d denote the Krull dimension of R. We are aiming to show that if R is a domain, then the integral closure R^+ of R in an algebraic closure of its fraction field is a big Cohen-Macaulay algebra for R. (This means that given any sequence of parameters in R, i.e., elements of the Jacobson radical that are part of a system of parameters in every local ring of R at a maximal ideal, then these elements form a regular sequence in R^+.)

Recall that an element c of R is a colon-killer if for every sequence of parameters $x_1, \ldots, x_{k+1}$ in R, c multiplies $(x_1, \ldots, x_k)R : x_{k+1}$ into $(x_1, \ldots, x_k)R$. The unmixed part of an ideal I is the intersection of its primary components for minimal primes, and is denoted I^{unmx}. A sequence of parameters $x, x_1, \ldots, x_k$ is called extraordinary if x and x_1 are colon-killers and for each i between 2 and k, the image of x_i is a colon-killer in $R/(x_1, \ldots, x_{i-1})R$.

The following results have been established already:

Theorem (existence of colon-killers). *If R is as above and $c \in R$ is such that R_c is Cohen-Macaulay, then c has a power c^N that kills all higher Koszul homology of sequences of parameters. In particular, c^N is a colon-killer.* $\square$

Theorem (equational forcing). *Let R be a ring of prime characteristic $p > 0$, let $x_1, \ldots, x_k$ be nonzerodivisors in R, and let $J = (z_1, \ldots, z_s)R$ be an ideal such that $J^{[p]} \subseteq I^{[p]} + (x_1 \cdots x_k)^{p-1}J$. Then there is a module-finite extension S of R such that $J \subseteq IS$. Moreover, if R is a domain, then S can be chosen to be a domain.*

The outline of the proof of the Cohen-Macaulay property of R^+ is as follows: We use the theorem on equational forcing to prove that Koszul relations on extraordinary parameters can be trivialized in a module-finite extension. (There is a recursive set-up involved). We use the theorem on existence of colon-killers to show that there are so many extraordinary systems of parameters that every Koszul relation on any sequence of parameters can be trivialized in a module-finite extension domain by doing so for a certain extraordinary sequence derived from the original one.

Recall that if I is an ideal of R and $x \in R$ we write $I :_R x^\infty$ for $\bigcup_t I_R x^t$, which is also the contraction of IR_x to R. Note that if x is a colon-killer and $x, x_1, \ldots, x_k$ is a sequence of parameters, then $(x_1^{a_1}, \ldots, x_k^{a_k})R :_R x^\infty = (x_1^{a_1}, \ldots, x_k^{a_k})R :_R x$ for any choice of positive integers $a_1, \ldots, a_k$.

The following two results are critical in showing that R^+ is a big Cohen-Macaulay algebra over the excellent local domain R in prime characteristic $p > 0$.

Theorem (unraveling for extraordinary parameters). *Let R be standard and let $x, x_1, \dots, x_k$ be an extraordinary sequence of parameters. Let $a_1, \dots, a_k$ be arbitrary positive integers. Then:*

$$(x_1^{a_1}, \dots, x_k^{a_k})R :_R x^\infty = \sum_{i=0}^{k-1}\Big(\prod_{\nu=1}^{i} x_\nu^{a_\nu - 1}\Big)\big((x_1, \dots x_i, x_{i+2}^{a_{i+2}}, \dots, x_n^{a_n})R :_R x^\infty\big)$$

$$+\Big(\prod_{\nu=1}^{k} x_\nu^{a_\nu - 1}\Big)\big((x_1, \dots, x_k)R :_R x^\infty\big).$$

We postpone the proof. Using this, we have already proved:

Critical Lemma. *Let R be a standard domain and let $x, x_1, \dots, x_k$ be an extraordinary sequence of parameters. Let $S = R^+$. Let $I = (x_1, \dots, x_k)R :_R x^\infty \subseteq R$. Suppose that every proper subset of $x, x_1, \dots, x_k$ forms a regular sequence in S. Then $I^{[p]}S \subseteq (x_1^p, \dots, x_k^p)S + (x_1 \cdots x_k)^{p-1}IS$. Hence, by the equational forcing lemma, $I \subseteq (x_1, \dots, x_k)S$.*[1]

The proof of the unraveling theorem is given by an iteration of the following result. We postponed the proof to give a discussion of the application of the result on the Cohen-Macaulayness of R^+ to prove the vanishing theorem for maps of Tor.[2] We restate the lemma needed now, this time providing the proof.

Lemma. *Let R be standard and let $x, x_1, \dots, x_h, y, z_{h+2}, \dots, z_k$ be a sequence of parameters such that*
(1) x is a colon-killer.
(2) The image of y is a colon-killer for R/I^{unmx}, where $I = (x_1, \dots, x_h)R$.
Let $I' = (z_{h+2}, \dots, z_k)R$. Then for every positive integer a,

$$(I + y^aR + I') :_R x^\infty = (I + I') :_R x^\infty + y^{a-1}\big((I + yR + I') :_R x^\infty\big).$$

(Recall that because x is a colon-killer, it does not matter with we write x^∞ or x in the denominators of the colon ideals.)

Proof. The two terms on the right are easily seen to be contained in the term on the left. Now suppose that $u \in (I + y^aR + I') :_R x^\infty$, so that we can write $x^N u = i + y^a r + i'$ where $i \in I$, $r \in R$, and $i' \in I'$. We pass to the ring $\overline{R} = R/I^{\mathrm{unmx}}$, using letters with bars over them to indicate images in $\overline{R}$. Then $\overline{x}^N\overline{u} = \overline{y}^a\overline{r} + \overline{i'}$ and, $\overline{u} \in (y^a, I')\overline{R} :_{\overline{R}} \overline{x}^N$. But $\overline{x}$, $\overline{y}$,

[1]Although the proof was given in an earlier lecture, it is short, and we repeat it here. It is easy to check that $I^{[p]} \subseteq (x_1^p, \dots, x_k^p) :_R x^\infty$. We apply the theorem on unraveling to write $(x_1^p, \dots, x_k^p) :_R x^\infty$ as a sum of several terms. In every term but the last there is a colon ideal involving a proper subset of the x's. These only become larger if computed in S, where these proper subsets are regular sequences. From this one sees that all the terms but the last are contained in $I^{[p]}S$. The last term is $(x_1 \cdots x_k)^{p-1}I$. Expanding to S yields the required result. $\square$

[2]This asserts that if $A \to R \to S$ are homomorphisms of Noetherian rings of prime characteristic $p > 0$ such that A and R are regular and R is a module-finite torsion-free extension of A, then for every A-module M, the map $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$ vanishes for $i \geq 1$.

and the $\overline{z}_j$ are still a sequence of parameters in $\overline{R}$ (the height of Jacobson radical decreases by h). Since $\overline{y} \in \overline{R}$ is a colon-killer by hypothesis, we have that $\overline{y}\,\overline{u} \in \overline{y}^a\overline{R} + I'\overline{R}$, and then $yu \in I^{\mathrm{unmx}} + y^aR + I'$. Choose $r \in R$ such that $yu - y^ar \in I^{\mathrm{unmx}} + I'$.

Next observe that $xI^{\mathrm{unmx}} \subseteq I$. To see this, fix a primary decomposition of I, and then choose w such that it is in every primary component corresponding to an embedded prime (the embedded primes have height $> h$) and such that $x_1, \dots, x_h, w$ is part of a system of parameters. This is an elementary prime avoidance argument. Then $I^{\mathrm{unmx}} = (x_1, \dots, x_h)R :_R w$, and the result now follows because x is a colon-killer. This yields $x(yu - y^ar) \in I + I'$ or $u - y^{a-1}r \in (I + I') :_R (xy)$. Since the generators of $I + I'$ together with xy also form a sequence of parameters, x multiplies $(I + I') :_R (xy)$ into $I + I'$, and so $u - y^{a-1}r \in (I + I') :_R x$. Thus, we need only show that $r \in (I + yR + I') :_R x$.

Now $y^{a-1}r = u + v$ where $v \in (I + I') :_R x \subseteq (I + y^aR + I') :_R x$, and u is in this latter ideal as well, so that $y^{a-1}r \in (I + y^aR + I') : x$ and $xy^{a-1}r \in I + y^aR + I'$. Choose $s \in R$ so that $xy^{a-1}r - y^as \in I + I'$. Then $xr - ys \in (I + I') :_R y^{a-1}$. Again, since x is a colon-killer, $x(xr - ys) \in I + I'$, which implies that $x^2r \in I + yR + I'$ and $r \in (I + yR + I') :_R x^2 = (I + yR + I') :_R x$ — this is the final application of the fact that x is a colon-killer in this argument. □

Proof of the unraveling theorem for extraordinary parameters. By repeated application of the Lemma we have:

$(x_1^{a_1}, \dots, x_k^{a_k}) :_R x^\infty \subseteq$ (applying the result with $y = x_1$)

$(x_2^{a_2}, \dots, x_k^{a_k}) :_R x^\infty + x_1^{a_1-1}\big((x_1, x_2^{a_2}, \dots, x_k^{a_k}) :_R x^\infty\big) \subseteq$

(applying the result with $y = x_2$ to the coefficient ideal of $x_1^{a_1-1}$)

$(x_2^{a_2}, \dots, x_k^{a_k}) :_R x^\infty + x_1^{a_1-1}\big((x_1, x_3^{a_3}, \dots, x_k^{a_k}) :_R x^\infty\big)$

$+x_1^{a_1-1}x_2^{a_2-1}\big(x_1, x_2, x_3^{a_3}, \dots, x_k^{a_k}) :_R x^\infty\big)$

Continuing in this way we have, at the hth step a sum containing h terms that will not change (i.e., these same terms will be present in all the further expressions), namely

$$\sum_{i=0}^{h-1}\Big(\prod_{\nu=1}^{i} x_\nu^{a_\nu-1}\Big)\big((x_1, \dots, x_i, x_{i+2}^{a_{i+2}}, \dots, x_k^{a_k})R :_R x^\infty\big),$$

and one term, which if $h < k$, will be expanded further using the Lemma, namely

$$\Big(\prod_{\nu=1}^{h} x_\nu^{a_\nu-1}\Big)\big((x_1, \dots, x_h, x_{h+1}^{a_{h+1}}, \dots, x_k^{a_k})R :_R x^\infty\big).$$

At the recursive step we use the Lemma to expand the colon ideal in large parenthesis that is on the right in the display just above, taking $y = x_{h+1}$, $a = a_{h+1}$, and z_s to be $x_s^{a_s}$ fpr $h + 2 \le s \le n$. The process terminates when $h = k$, giving the formula of the Theorem on unraveling. □

Note that we have now completed the proof of the Critical Lemma.

Our next objective is to show that getting rid of relations on extraordinary sequences of parameters is sufficient to get rid of all relations on all sequences of parameters. Towards this end, we define a sequence of parameters $\underline{y} = y_0, \ldots, y_k$ to be *derived* from a sequence of parameters $\underline{x} = x_0, \ldots, x_k$ if it can be obtained from it by a finite sequence of operations of the following three kinds, which we refer to as *permissible* operations:

(1) Permuting the x_i.

(2) Replacing x_i by $x_i + rx_j$ for some $j \neq i$ and some $r \in R$, while leaving the other x_s unchanged.

(3) Replacing x_i by $x_i y$, where y is an element of the Jacobson radical of R such that the elements $x_0, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k$ is also a sequence of parameters. (Note that this implies that $x_0, \ldots, x_{i-1}, x_i y, x_{i+1}, \ldots, x_k$ is also a sequence of paramters.)

Observe that the relation $\underline{y}$ is derived from $\underline{x}$ is reflexive and transitive but that it is *not* symmetric: operations of types (1) and (2) do not change the ideal generated by the parameters, but operations of type (3) result in a strict decrease of that ideal. Note also that with an operation of type (3) we may replace x_i by a power of x_i. The relevance of this notion is shown by the following:

Proposition. *Let R be standard and $\underline{y}$ derived from $\underline{x}$ of length $k+1$ as above. Let $S = R^+$. Suppose that every sequence of parameters of length k in R is a regular sequence in S. Then if all the relations on the y_j become trivial in S, the same is true of the x_j. That is, if the map $H_1(\underline{y};\, R) \to H_1(\underline{y};\, S)$ is zero, then the map $H_1(\underline{x};\, R) \to H_1(\underline{x};\, S)$ is zero.*

Proof. We proceed by induction on the number of operations need to derive $\underline{y}$ from $\underline{x}$. Evidently, we may reduce at once to the case of just one operation. Since operations of types (1) and (2) don't change the Kozsul homology, we need only consider a single operation of type (3). From the symmetry of Kozsul homology in the elements of the sequence it suffices to consider the case where $i = k$, so that the new sequence is $x_0, x_1, \ldots, x_{k-1}, yx_k$. Suppose that we have a relation $\sum_{j=0}^{k-1} r_j x_j + r x_k = 0$ with the r_j and r in R. We multiply by y to get $\sum_{j=0}^{k-1} (r_j y) x_j + r(x_k y) = 0$, which is a relation on $x_0, x_1, \ldots, x_{k-1}, yx_k$ over R and so trivial over S, that is, it is an S-linear combination of standard Koszul relations. This implies that $r \in (x_0, \ldots, x_{k-1})S$, which enables us to subtract an S-linear combination of standard Koszul relations on $x_0, \ldots, x_k$ involving x_k and one of the x_j for $j < k$ to get a relation over S on $x_0, \ldots, x_{k-1}$, because the coefficient of x_k is now 0. But these elements form a regular sequence in S, and so this relation is trivial over S. $\square$