Commutative Algebra Seminar: Lecture of January 16, 2003

We need one more major lemma before we can prove the Cohen-Macaulay property for R^+. We need two preliminary results.

Lemma. *Let R be any ring.*

(a) *Let $x, y \in R$ and let $P_1, \ldots, P_s$ be finitely many primes of R such that $x + Ry = \{x + ry : r \in R\} \subseteq \bigcup_j P_j$. Then for some j, both x and y are in P_j.*

(b) *Let R be standard. Suppose that $x_0, \ldots, x_k$ and $x_0, \ldots, x_i, z$ are both sequences of parameters in the Jacobson radical, $-1 \le i \le k$. Then there exist $r_j \in R$, $i+2 \le j \le k$ such that*

$$x_0, \ldots, x_i, z, x_{i+2} + r_{i+2}x_{i+1}, \ldots, x_k + r_k x_{i+1}$$

is a sequence of parameters. Hence,

$$x_0, \ldots, x_i, zx_{i+1}, x_{i+2} + r_{i+2}x_{i+1}, \ldots, x_k + r_k x_{i+1}$$

is a sequence of parameters derived from $x_0, \ldots, x_k$ by permissible operations, and if z is a colon-killer (or a colon-killer modulo $\mathfrak{A}$), then zx_{i+1} is as well.

Proof. (a) This part is immediate from the prime avoidance result Theorem 124 on p. 90 of [I. Kaplansky, *Commutative Algebra*, Allyn and Bacon, Boston, 1970; rev. ed. Univ. of Chicago Press, Chicago, 1974].

(b) Assume that the r_j have been chosen for $j < h$ such that $x_0, \ldots, x_i, z$ together with the elements $x_j + r_j x_{i+1}$ for $i+2 \le j < h$ are a sequence of parameters. Note that we allow $h = i+2$ to begin the induction. Then all we need to do to complete the induction is to prove that we can choose r_h so that $x_h + rx_{i+1}$ is not in any minimal prime P of the ideal generated by the h elements already determined. By part (a), this will be possible unless both x_h and x_{i+1} are in one of those minimal primes. But then all of $x_0, \ldots, x_h$ will be in that prime, and so it must have height at least $h+1$, contradicting the assumption that it is a minimal prime of an ideal generated by h elements.

For the final statement, note that $x_0, \ldots, x_{i+1}, x_{i+2} + r_{i+2}x_{i+1}, \ldots, x_k + r_k x_{i+1}$ is clearly a sequence of parameters derived from $x_0, \ldots, x_k$, and we get the stated conclusion by one permissible operation of type (3), given the statement of the second sentence. Since a multiple of a colon-killer is obviously a colon-killer, the final part is obvious. □

Lemma. *Let R be standard.*[1]

(a) *The Cohen-Macaulay locus is open.*

(b) *If (0) has no embedded primes in R, then the ideal defining the non-Cohen-Macaulay locus in R has height at least 2.*

Proof. (a) First consider the case where R is a complete local ring. Since R is equidimensional, it can be written as T/I where T is regular and I is an ideal all of whose

[1]Both parts of this result are valid under the much weaker hypothesis that R be excellent.

minimal primes have the same height h. From the lemma giving the Ext characterization of the Cohen-Macaulay property in the notes from the Lecture of December 2, R_Q is not Cohen-Macaulay if and only if $Q \supseteq I$ is in the union of the supports of the modules $\mathrm{Ext}^t_T(T/I,\, T)$ for $t \neq h$, which is a closed set.

Now consider the general case. The completion $\widehat{R}$ of R with respect to its Jacobson radical will be a finite product of complete equidimensional local rings: thus, Spec $\widehat{R}$ is the disjoint union of the Specs of these, and it follows that the Cohen-Macaulay locus U for $\widehat{R}$ is open.

Next note that if $R \to S$ is a faithfully flat map of Noetherian rings, then $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is onto. Moreover, a subset of $\mathrm{Spec}\, R$ is closed if and only if its inverse image in $\mathrm{Spec}\, S$ is closed: we refer to (6.H), Theorem 7, and the Remark in between on pp. 46–7 of [H. Matsumura, *Commutative Algebra*, W. A. Benjamin, Inc., New York, 1970]. Now, when Q lies over P, R_P is Cohen-Macaulay iff $\widehat{R}_Q$ is Cohen-Macaulay, since the map is flat and the closed fiber is Cohen-Macaulay (even regular). It follows that a prime P in R is such that R_P is Cohen-Macaulay (respectively, not Cohen-Macaulay) if and only if for some (equivalently, every) prime Q of $\widehat{R}$ lying over P, $\widehat{R}_Q$ is (respectively, is not) Cohen-Macaulay.

(b) Let I be the ideal defining the non-Cohen-Macaulay locus. If I has height at most one, we may localize at a minimal prime of I to obtain a local non-Cohen-Macaulay ring of dimension at most one in which (0) has no embedded prime, a contradiction. □

The last of the major preliminaries is:

Theorem (Moving Lemma). *Let R be a standard domain and let $\underline{x} = x_0, \ldots, x_k$ be a sequence of parameters in the Jacobson radical. Then there is a derived sequence of parameters $y_0, \ldots, y_k$ that is extraordinary.*

Proof. If R has dimension at most one it is Cohen-Macaulay and there is nothing to prove. Henceforth we assume that $\dim R \geq 2$. Let I be the defining ideal of the non-Cohen-Macaulay locus, which by the Lemma just above has height at least 2.

Choose a parameter z in I. By the theorem on the existence of colon-killers, we may replace z by a power that is a colon-killer. If $k = 0$ we replace x_0 by $x_0 z$ and we are done. Assume that $k \geq 1$. We shall successively derive sequences of parameters from $\underline{x}$ in such a way that increasingly longer initial segments satisfy the condition to be an extraordinary sequence of parameters.

By part (b) of the first Lemma above we may choose $r_1, \ldots, r_k \in R$ such that $x_0 z,\, x_1 + r_1 x_0, \ldots, x_k + r_k x_0$ is derived from $\underline{x}$, and $x_0 z$ is a colon-killer. We change notation and write $y_0, x_1, \ldots, x_k$ for this derived sequence of parameters.

Next, choose z in I (and in the Jacobson radical) such that y_0, z is a sequence of parameters. We can do this because I has height at least two. Replace z by a power, if necessary, so that z is a colon-killer. We may again apply part (b) of the Lemma above to choose $r_2, \ldots, r_k \in R$ so that $y_0,\, x_1 z,\, x_2 + r_2 x_1, \ldots, x_k + r_2 x_1$ is a sequence of parameters derived from $\underline{x}$ such that y_0 and $y_1 = x_1 z$ are colon-killers.

Now suppose that for a certain integer h, $1 \leq h < k$, we have a derived sequence of parameters such that the first $h+1$ terms give an extraordinary sequence of parameters. Call the sequence $y_0, y_1, \ldots, y_h, x_{h+1}, \ldots, x_k$ (this is a change of notation — the x_j are not the original ones). We shall show that there is a sequence derived from this one in which the first $h+1$ terms $y_1, \ldots, y_h$ do not change, and the next term $y = y_{h+1}$ is a colon-killer in $R/\mathfrak{A}$, where $\mathfrak{A} = (y_1, \ldots, y_h)^{\mathrm{unmx}}$. This will complete the inductive proof of the result.

Now, $\dim R \geq k+1 \geq h+2$, and so $\dim R/\mathfrak{A} \geq 2$. Since $R/\mathfrak{A}$ is unmixed, the defining ideal $\mathfrak{B}/\mathfrak{A}$ of the non-Cohen-Macaulay locus has height at least 2, where $\mathfrak{B}$ is its inverse image in R, and so $\mathfrak{B}$ has height at least $h+2$ in R. Consequently, there is an element $z \in \mathfrak{B}$ not in the union of the minimal primes of $(y_0, \ldots, y_h)R$. Replacing z by a power we see that we may assume that the image of z is a colon-killer in $R/\mathfrak{A}$. Again, by an application of part (b) of the first Lemma, we can choose $r_{h+2}, \ldots, r_k \in r$ such that

$$y_0, \ldots, y_h, zx_{h+1}, x_{h+2} + r_{h+2}x_{h+1}, \ldots, x_k + r_k x_{h+1}$$

is derived from $y_0, \ldots, y_h, x_{h+1}, \ldots, x_k$, and, evidently, the image of zx_{h+1} is a colon-killer in $R/\mathfrak{A}$. $\square$