Commutative Algebra Seminar: Lecture of January 23, 2003

We restate and prove the main theorem that we have been aiming for:

Theorem. *Let R be a semi-local bi-equidimensional excellent domain of characteristic $p > 0$. Then R^+ is a big Cohen-Macaulay algebra for R: every sequence of parameters in the Jacobson radical of R is a regular sequence in R^+.*

Proof. First note that the hypothesis is preserved by taking a module-finite extension domain: the fact that the ring is still bi-equidimensional follows, for example, from (14.C) Theorem 23 on p. 84 of [H. Matsumura, *Commutative Algebra*, W. A. Benjamin, Inc., New York, 1970]. We use induction on the length k of the sequence of parameters. If $k = 1$ the result is obvious. If $k = 2$ we use the fact that the normalization of such a ring R is module-finite over R (R is excellent), and it follows that R^+ is a directed union of normal semi-local domains, and in such a domain a sequence of parameters of length 2 is a regular sequence.

We now assume that for a certain $k \geq 2$ and for all R, every sequence of parameters of length at most k is a regular sequence in R^+. We want to prove this for sequences $x_0, \ldots, x_k$ of length $k+1$. Suppose that $sx_k = \sum_{i=0}^{k-1} s_i x_i$ with $s, s_0, \ldots, s_{k-1} \in R^+$. Choose a module-finite extension R_1 of R with $s, s_0, \ldots, s_{k-1} \in R_1$. We may change notation and write R for R_1. Thus, it suffices to show that if $rx_k = \sum_{i=0}^{k-1} r_i x_i$ with $r, r_0, \ldots, r_{k-1} \in R$, then $r \in (x_0, \ldots, x_{k-1})R^+$. This will follow if the relation $(r_0, \ldots, r_{k-1}, -r_k)$ is trivial over R^+. By the Moving Lemma there is a sequence of parameters $y_0, \ldots, y_k$ derived from $x_0, \ldots, x_k$ by permissible operations such that $y_0, \ldots, y_k$ is extraordinary. By the Critical Lemma every relation on $y_0, \ldots, y_k$ is trivial over R^+. By the Proposition proved at the end of the notes on the Lecture of January 9, this implies that the original relation on $x_0, \ldots, x_k$ is trivial over R^+. $\square$

Remark. In the paper [M. Hochster and C. Huneke, *Infinite integral extensions and big Cohen-Macaulay algebras*, Annals of Math. **135** (1992), 53–89] where this result is proved, a graded analogue of this result is obtained. Let R be an $\mathbb{N}$-graded domain finitely generated over a field K with $R_0 = K$. Let $\mathbb{Q}_{\geq 0}$ denote the nonnegative rational numbers. Then there is a maximum $\mathbb{Q}_{\geq 0}$-graded integral extension domain $R^{+\mathrm{GR}}$ of R, consisting of elements θ of R^+ such that the ideal of polynomials in $R[z]$ satisfied by $z = \theta$ is graded if one assigns a degree to z in a suitable way. If $\theta \neq 0$ it will satisfy a homogeneous monic polynomial $f(z) \in R[z]$ with a nonzero constant term, and then the degree of θ will be the degree of the constant term divided by the degree of f. Within this $\mathbb{Q}_{\geq 0}$-graded ring is an $\mathbb{N}$-graded R-subalgebra $R^{+\mathrm{gr}}$ which is the direct sum of the graded components indexed by elements of $\mathbb{N}$. This is a maximum $\mathbb{N}$-graded module-finite extension of R, and is unique up to non-unique isomorphism of $\mathbb{N}$-graded R-algebras. (For example, if x and y are degree one elements of R, the equation $z^2 - (x^6 + y^6)$ will have a degree 3 root in $R^{+\mathrm{gr}}$, as will $z^2 + (x^3 + y^3)z - (x^6 + x^5y - y^6)$, while $z^2 - (x^5 + y^5)$ will have a root in the larger $\mathbb{Q}_{\geq 0}$-graded ring $R^{+\mathrm{GR}}$ but not in $R^{+\mathrm{gr}}$.)

Theorem. *With notation and terminology as in the paragraph just preceding, every homogeneous system of parameters in R is a regular sequence in $R^{+\mathrm{gr}}$.*

The proof is essentially the same as the one we gave for the case handled here: one needs to do some extra work, keeping track of the grading at every step in the argument.

This means that every relation on homogeneous parameters in an $\mathbb{N}$-graded domain R as above becomes trivial in a suitable $\mathbb{N}$-graded module-finite extension domain of R.

The larger $\mathbb{Q}_{\geq 0}$-graded ring $R^{+\mathrm{GR}}$ also has the property that homogeneous sequences of parameters are regular sequences: for each n, one may form the subring which is the direct sum of the graded pieces that have degree that is an integer multiple of $1/n!$, which, after scaling the grading, has the same form as R^+, and so has the requisite property. But $R^{+\mathrm{GR}}$ is the increasing union of these. On the other hand, if one knows instead that $R^{+\mathrm{GR}}$ has the required property, one gets it for $R^{+\mathrm{gr}}$ as well: $R^{+\mathrm{gr}}$ is a direct summand of $R^{+\mathrm{GR}}$ via the map that kills all forms whose degree is not an integer, while fixing forms whose degree is an integer.

We next discuss some applications of the result just proved. We have done one already: the vanishing theorem for maps of Tor. However, we shall give a second proof of this theorem. We first want to define the plus closure of a submodule N of M over any Noetherian ring R of characteristic p. If R is a domain, we define $u \in M$ to be in $N^+ = N^+_M$ if u is in the submodule of $R^+ \otimes_R M$ spanned by the image of N. We extend this definition to arbitrary R by defining u to be in N^+ if the image of u in M/PM is in the plus closure of the image of N (equivalently, N/PN) in M/PM for every minimal prime P of R. If $M/N \cong M'/N'$ and the image of $u \in M$ corresponds to the image of $u' \in M'$, then $u \in N^+_M$ iff $u' \in N'^+_{M'}$. (Of course, tight closure has the analogous property.) Thus, if R is a domain we can test whether an element is in a plus closure by mapping a free module G on to M, and replacing N by the inverse image H of N in G, and u by some element $g \in G$ that maps to u. G is a direct sum of copies of R, and $R^+ \otimes_R G$ is a direct sum of copies of R^+. The span of the image of H in $R^+ \otimes G$ is much like the expansion of an ideal $I \subseteq R$ to R^+, which is the case where $G = R$ and $H = I$. By an argument that is the same as what was done for the ideal case earlier, $N \subseteq N^+ \in N^*$ (in M).

Conjecture. *If R is a locally excellent Noetherian ring of positive prime characteristic and $N \subseteq M$ are finitely generated modules, then $N^+ = N^*$.*

This is open even in dimension 2, even for ideals in rings of the form

$$K[x, y, z]/(x^3 + y^3 + z^3)$$

where K is a field of positive characteristic $\neq 3$. (It is known for homogeneous m-primary ideals in such rings, by results of Holger Brenner (cf. [H. Brenner, *Tight closure and projective bundles*, preprint] and [H. Brenner, *Tight closure and plus closure for cones over elliptic curves*, preprint].) We shall soon be working on settling certain cases of this conjecture: in particular, the result for ideals generated by part of a system of parameters.

One reason that the conjecture is important is that plus closure is easily seen to commute with localization. (This reduces to the case where the ring is a domain. Let W be a multiplicative system, and $H \subseteq G$, where G is free. Let $S = R^+$. Let $\_{W}$ indicate localization at W. Let $GS = S \otimes_R G$ and HS be the span of H in GS, with similar conventions for S_W. We have already observed that $(R_W)^+ \cong S_W$. Thus, $(H_W)^+ = \bigl(H(R_W)^+\bigr) \cap G_W = (HS_W) \cap G_W = (HS \cap G)_W = (H^+)_W$, as required.)

If α is a linear map of free modules (or a matrix) over R we write $I_r(\alpha)$ for the ideal generated by the $r \times r$ minors (subdeterminants) of a matrix for α: this was shown earlier to be independent of the choice of matrix for the linear map. If $r = 0$, this is defined to be the unit ideal.

We next observe the following:

Theorem (phantom acyclicity criterion). *Let R be a reduced locally excellent Noetherian ring. Let $G_\bullet$ denote a finite free complex over R, say*

$$0 \to R^{b_n} \to \cdots \to R^{b_1} \to R^{b_0} \to 0.$$

Let α_i denote the matrix of the map $G_i \to G_{i-1}$, $1 \le i \le n$. Let r_i denote the determinantal rank of α_i. Suppose that the following two conditions hold:
(1) $b_i = r_i + r_{i-1}$, $1 \le i \le n$.
(2) *Modulo every minimal prime of R, the height of $I_{r_i}(\alpha_i) \ge i$, $1 \le i \le n$.*
Then for every $i \ge 1$ if B_i is the image of α_{i+1} and Z_i is the kernel of α_i, then $Z_i \subseteq B_i^*$. In fact, $Z_i \subseteq B_i^+$.

When a complex is such that the cycles at a certain spot are in the tight closure of the boundaries in the ambient module of the complex, we say the complex has *phantom homology* (or *phantom cohomology*, if appropriate) at that spot. Thus, the theorem asserts that if finite free complexes satisfying conditions (1) and (2) have phantom higher homology.

Proof. Because the ideals of rank size minors all have positive height modulo any minimal prime, if we pass to R/P for P minimal (i.e., tensor the complex with R/P over R) the determinantal ranks of the maps do not change, and the hypotheses are preserved. We may therefore assume without loss of generality that R is a domain. The stated result will now follow if $R^+ \otimes_R G_\bullet$ is acyclic: each cycle will be in the boundaries after expansion to R^+.

This issue of acyclicity is local on R. When we replace R by one of its local rings, the determinantal ranks do not change. The heights of rank size minors can only increase. Moreover, $(R^+)_P \cong (R_P)^+$. We have therefore reduced to the case where R is excellent and local.

But then $I_{r_i}(\alpha_i)$, since it has height at least i, contains a sequence of parameters of length i, which will be a regular sequence in R^+. Acyclicity now follows from the acyclicity Theorem in the Lecture Notes from November 25. $\square$